

# General Relativity Today<sup>\*†</sup>

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**Abstract:** After recalling the conceptual foundations and the basic structure of general relativity, we review some of its main modern developments (apart from cosmology) : (i) the post-Newtonian limit and weak-field tests in the solar system, (ii) strong gravitational fields and black holes, (iii) strong-field and radiative tests in binary pulsar observations, (iv) gravitational waves, (v) general relativity and quantum theory.

## 1 Introduction

The *theory of general relativity* was developed by Einstein in work that extended from 1907 to 1915. The starting point for Einstein's thinking was the composition of a review article in 1907 on what we today call the *theory of special relativity*. Recall that the latter theory sprang from a new kinematics governing length and time measurements that was proposed by Einstein in June of 1905 [1], [2], following important pioneering work by Lorentz and Poincaré. The theory of special relativity essentially poses a new fundamental framework (in place of the one posed by Galileo, Descartes, and Newton) for the formulation of physical laws: this framework being the chrono-geometric space-time structure of Poincaré and Minkowski. After 1905, it therefore seemed a natural task to formulate, reformulate, or modify the then known physical laws so that they fit within the framework of special relativity. For Newton's law of gravitation, this task was begun (before Einstein had even supplied his conceptual crystallization in 1905) by Lorentz (1900) and Poincaré (1905), and was pursued in the period from 1910 to 1915 by Max Abraham, Gunnar Nordström and Gustav Mie (with these latter researchers developing *scalar* relativistic theories of gravitation).

Meanwhile, in 1907, Einstein became aware that gravitational interactions possessed particular characteristics that suggested the necessity of *generalizing* the framework and structure of the 1905 theory of relativity. After many years of intense intellectual effort, Einstein succeeded in constructing a *generalized*

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*theory of relativity* (or *general relativity*) that proposed a profound modification of the chrono-geometric structure of the space-time of special relativity. In 1915, in place of a simple, neutral arena, given a priori, independently of all material content, space-time became a physical “field” (identified with the gravitational field). In other words, it was now a dynamical entity, both influencing and influenced by the distribution of mass-energy that it contains.

This radically new conception of the structure of space-time remained for a long while on the margins of the development of physics. Twentieth century physics discovered a great number of new physical laws and phenomena while working with the space-time of special relativity as its fundamental framework, as well as imposing the respect of its symmetries (namely the Lorentz-Poincaré group). On the other hand, the theory of general relativity seemed for a long time to be a theory that was both poorly confirmed by experiment and without connection to the extraordinary progress springing from application of quantum theory (along with special relativity) to high-energy physics. This marginalization of general relativity no longer obtains. Today, general relativity has become one of the essential players in cutting-edge science. Numerous high-precision experimental tests have confirmed, in detail, the pertinence of this theory. General relativity has become the favored tool for the description of the macroscopic universe, covering everything from the big bang to black holes, including the solar system, neutron stars, pulsars, and gravitational waves. Moreover, the search for a consistent description of fundamental physics in its entirety has led to the exploration of theories that unify, within a general quantum framework, the description of matter and all its interactions (including gravity). These theories, which are still under construction and are provisionally known as string theories, contain general relativity in a central way but suggest that the fundamental structure of space-time-matter is even richer than is suggested separately by quantum theory and general relativity.

## 2 Special Relativity

We begin our exposition of the theory of general relativity by recalling the chrono-geometric structure of space-time in the theory of *special* relativity. The structure of Poincaré-Minkowski space-time is given by a generalization of the Euclidean geometric structure of ordinary space. The latter structure is summarized by the formula  $L^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$  (a consequence of the Pythagorean theorem), expressing the square of the distance  $L$  between two points in space as a sum of the squares of the differences of the (orthonormal) coordinates  $x, y, z$  that label the points. The symmetry group of Euclidean geometry is the group of coordinate transformations  $(x, y, z) \rightarrow (x', y', z')$  that leave the quadratic form  $L^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$  invariant. (This group is generated by translations, rotations, and “reversals” such as the transformation given by reflection in a mirror, for example:  $x' = -x, y' = y, z' = z$ .)

The Poincaré-Minkowski space-time is defined as the ensemble of *events* (idealizations of what happens at a particular point in space, at a particular moment

in time), together with the notion of a (*squared*) *interval*  $S^2$  defined between any two events. An event is fixed by four coordinates,  $x, y, z$ , and  $t$ , where  $(x, y, z)$  are the spatial coordinates of the point in space where the event in question “occurs,” and where  $t$  fixes the instant when this event “occurs.” Another event will be described (within the same reference frame) by four different coordinates, let us say  $x + \Delta x$ ,  $y + \Delta y$ ,  $z + \Delta z$ , and  $t + \Delta t$ . The points in space where these two events occur are separated by a distance  $L$  given by the formula above,  $L^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ . The moments in time when these two events occur are separated by a time interval  $T$  given by  $T = \Delta t$ . The squared interval  $S^2$  between these two events is given as a function of these quantities, by definition, through the following generalization of the Pythagorean theorem:

$$S^2 = L^2 - c^2 T^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2 (\Delta t)^2, \quad (1)$$

where  $c$  denotes the speed of light (or, more precisely, the maximum speed of signal propagation).

Equation (1) defines the *chrono-geometry* of Poincaré-Minkowski space-time. The symmetry group of this chrono-geometry is the group of coordinate transformations  $(x, y, z, t) \rightarrow (x', y', z', t')$  that leave the quadratic form (1) of the interval  $S$  invariant. We will show that this group is made up of linear transformations and that it is generated by translations in space and time, spatial rotations, “boosts” (meaning special Lorentz transformations), and reversals of space and time.

It is useful to replace the time coordinate  $t$  by the “light-time”  $x^0 \equiv ct$ , and to collectively denote the coordinates as  $x^\mu \equiv (x^0, x^i)$  where the Greek indices  $\mu, \nu, \dots = 0, 1, 2, 3$ , and the Roman indices  $i, j, \dots = 1, 2, 3$  (with  $x^1 = x$ ,  $x^2 = y$ , and  $x^3 = z$ ). Equation (1) is then written

$$S^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu, \quad (2)$$

where we have used the Einstein summation convention<sup>1</sup> and where  $\eta_{\mu\nu}$  is a diagonal matrix whose only non-zero elements are  $\eta_{00} = -1$  and  $\eta_{11} = \eta_{22} = \eta_{33} = +1$ . The symmetry group of Poincaré-Minkowski space-time is therefore the ensemble of Lorentz-Poincaré transformations,

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu, \quad (3)$$

where  $\eta_{\alpha\beta} \Lambda^\alpha_\mu \Lambda^\beta_\nu = \eta_{\mu\nu}$ .

The chrono-geometry of Poincaré-Minkowski space-time can be visualized by representing, around each point  $x$  in space-time, the locus of points that are separated from the point  $x$  by a unit (squared) interval, in other words the ensemble of points  $x'$  such that  $S^2_{xx'} = \eta_{\mu\nu} (x'^\mu - x^\mu)(x'^\nu - x^\nu) = +1$ . This locus is a one-sheeted (unit) hyperboloid.

If we were within an ordinary Euclidean space, the ensemble of points  $x'$  would trace out a (unit) sphere centered on  $x$ , and the “field” of these spheres

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<sup>1</sup>Every repeated index is supposed to be summed over all of its possible values.

centered on each point  $x$  would allow one to completely characterize the Euclidean geometry of the space. Similarly, in the case of Poincaré-Minkowski space-time, the “field” of unit hyperboloids centered on each point  $x$  is a visual characterization of the geometry of this space-time. See Figure 1. This figure gives an idea of the symmetry group of Poincaré-Minkowski space-time, and renders the rigid and homogeneous nature of its geometry particularly clear.

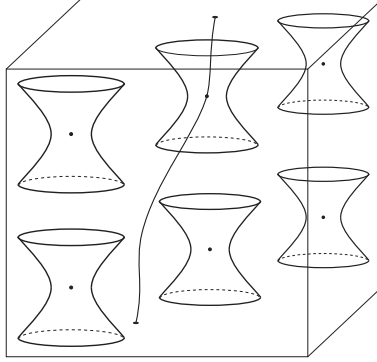


Figure 1: Geometry of the “rigid” space-time of the theory of special relativity. This geometry is visualized by representing, around each point  $x$  in space-time, the locus of points separated from the point  $x$  by a unit (squared) interval. The space-time shown here has only three dimensions: one time dimension (represented vertically),  $x^0 = ct$ , and two spatial dimensions (represented horizontally),  $x, y$ . We have also shown the ‘space-time line’, or ‘world-line’, (moving from the bottom to the top of the “space-time block,” or from the past towards the future) representing the history of a particle’s motion.

The essential idea in Einstein’s article of June 1905 was to impose the group of transformations (3) as a symmetry group of the fundamental laws of physics (“the principle of relativity”). This point of view proved to be extraordinarily fruitful, since it led to the discovery of new laws and the prediction of new phenomena. Let us mention some of these for the record: the relativistic dynamics of classical particles, the dilation of lifetimes for relativistic particles, the relation  $E = mc^2$  between energy and inertial mass, Dirac’s relativistic theory of quantum spin  $\frac{1}{2}$  particles, the prediction of antimatter, the classification of particles by rest mass and spin, the relation between spin and statistics, and the CPT theorem.

After these recollections on special relativity, let us discuss the special feature of gravity which, in 1907, suggested to Einstein the need for a profound generalization of the chrono-geometric structure of space-time.

### 3 The Principle of Equivalence

Einstein’s point of departure was a striking experimental fact: all bodies in an external gravitational field fall with the same acceleration. This fact was pointed out by Galileo in 1638. Through a remarkable combination of logical reasoning, thought experiments, and real experiments performed on inclined planes,<sup>2</sup> Galileo was in fact the first to conceive of what we today call the “universality of free-fall” or the “weak principle of equivalence.” Let us cite the conclusion that Galileo drew from a hypothetical argument where he varied the ratio between the densities of the freely falling bodies under consideration and the resistance of the medium through which they fall: “Having observed this I came to the conclusion that in a medium totally devoid of resistance all bodies would fall with the same speed” [3]. This universality of free-fall was verified with more precision by Newton’s experiments with pendulums, and was incorporated by him into his theory of gravitation (1687) in the form of the identification of the inertial mass  $m_i$  (appearing in the fundamental law of dynamics  $\mathbf{F} = m_i \mathbf{a}$ ) with the gravitational mass  $m_g$  (appearing in the gravitational force,  $F_g = G m_g m'_g / r^2$ ):

$$m_i = m_g . \quad (4)$$

At the end of the nineteenth century, Baron Roland von Eötvös verified the equivalence (4) between  $m_i$  and  $m_g$  with a precision on the order of  $10^{-9}$ , and Einstein was aware of this high-precision verification. (At present, the equivalence between  $m_i$  and  $m_g$  has been verified at the level of  $10^{-12}$  [4].) The point that struck Einstein was that, given the precision with which  $m_i = m_g$  was verified, and given the equivalence between inertial mass and energy discovered by Einstein in September of 1905 [2] ( $E = m_i c^2$ ), one must conclude that all of the various forms of energy that contribute to the inertial mass of a body (rest mass of the elementary constituents, various binding energies, internal kinetic energy, etc.) do contribute in a strictly identical way to the gravitational mass of this body, meaning both to its capacity for reacting to an external gravitational field and to its capacity to create a gravitational field.

In 1907, Einstein realized that the equivalence between  $m_i$  and  $m_g$  implicitly contained a deeper equivalence between inertia and gravitation that had important consequences for the notion of an inertial reference frame (which was a fundamental concept in the theory of special relativity). In an ingenious thought experiment, Einstein imagined the behavior of rigid bodies and reference clocks within a freely falling elevator. Because of the universality of free-fall, all of the objects in such a “freely falling local reference frame” would appear not to be accelerating with respect to it. Thus, with respect to such a reference frame, the exterior gravitational field is “erased” (or “effaced”). Einstein therefore postulated what he called the “principle of equivalence” between gravitation and inertia. This principle has two parts, that Einstein used in turns. The first part says that, for any external gravitational field whatsoever, it is possible to

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<sup>2</sup>The experiment with falling bodies said to be performed from atop the Leaning Tower of Pisa is a myth, although it aptly summarizes the essence of Galilean innovation.

locally “erase” the gravitational field by using an appropriate freely falling local reference frame and that, because of this, the non-gravitational physical laws apply within this local reference frame just as they would in an inertial reference frame (free of gravity) in special relativity. The second part of Einstein’s equivalence principle says that, by starting from an inertial reference frame in special relativity (in the absence of any “true” gravitational field), one can create an apparent gravitational field in a local reference frame, if this reference frame is accelerated (be it in a straight line or through a rotation).

## 4 Gravitation and Space-Time Chrono-Geometry

Einstein was able (through an extraordinary intellectual journey that lasted eight years) to construct a new theory of gravitation, based on a rich generalization of the 1905 theory of relativity, starting just from the equivalence principle described above. The first step in this journey consisted in understanding that the principle of equivalence would suggest a profound modification of the chrono-geometric structure of Poincaré-Minkowski space-time recalled in Equation (1) above.

To illustrate, let  $X^\alpha$ ,  $\alpha = 0, 1, 2, 3$ , be the space-time coordinates in a local, freely-falling reference frame (or *locally inertial reference frame*). In such a reference frame, the laws of special relativity apply. In particular, the infinitesimal space-time interval  $ds^2 = dL^2 - c^2 dT^2$  between two neighboring events within such a reference frame  $X^\alpha$ ,  $X'^\alpha = X^\alpha + dX^\alpha$  (close to the center of this reference frame) takes the form

$$ds^2 = dL^2 - c^2 dT^2 = \eta_{\alpha\beta} dX^\alpha dX^\beta, \quad (5)$$

where we recall that the repeated indices  $\alpha$  and  $\beta$  are summed over all of their values ( $\alpha, \beta = 0, 1, 2, 3$ ). We also know that in special relativity the local energy and momentum densities and fluxes are collected into the ten components of the *energy-momentum tensor*  $T^{\alpha\beta}$ . (For example, the energy density per unit volume is equal to  $T^{00}$ , in the reference frame described by coordinates  $X^\alpha = (X^0, X^i)$ ,  $i = 1, 2, 3$ .) The conservation of energy and momentum translates into the equation  $\partial_\beta T^{\alpha\beta} = 0$ , where  $\partial_\beta = \partial/\partial X^\beta$ .

The theory of special relativity tells us that we can change our locally inertial reference frame (while remaining in the neighborhood of a space-time point where one has “erased” gravity) through a Lorentz transformation,  $X'^\alpha = \Lambda^\alpha_\beta X^\beta$ . Under such a transformation, the infinitesimal interval  $ds^2$ , Equation (5), remains invariant and the ten components of the (symmetric) tensor  $T^{\alpha\beta}$  are transformed according to  $T'^{\alpha\beta} = \Lambda^\alpha_\gamma \Lambda^\beta_\delta T^{\gamma\delta}$ . On the other hand, when we pass from a *locally* inertial reference frame (with coordinates  $X^\alpha$ ) to an *extended* non-inertial reference frame (with coordinates  $x^\mu$ ;  $\mu = 0, 1, 2, 3$ ), the transformation connecting the  $X^\alpha$  to the  $x^\mu$  is no longer a *linear* transformation (like the Lorentz transformation) but becomes a *non-linear* transformation  $X^\alpha = X^\alpha(x^\mu)$  that can take any form whatsoever. Because of this, the value of

the infinitesimal interval  $ds^2$ , when expressed in a general, extended reference frame, will take a more complicated form than the very simple one given by Equation (5) that it had in a reference frame that was locally in free-fall. In fact, by differentiating the non-linear functions  $X^\alpha = X^\alpha(x^\mu)$  we obtain the relation  $dX^\alpha = \partial X^\alpha / \partial x^\mu dx^\mu$ . By substituting this relation into (5) we then obtain

$$ds^2 = g_{\mu\nu}(x^\lambda) dx^\mu dx^\nu, \quad (6)$$

where the indices  $\mu, \nu$  are summed over  $0, 1, 2, 3$  and where the ten functions  $g_{\mu\nu}(x)$  (symmetric over the indices  $\mu$  and  $\nu$ ) of the four variables  $x^\lambda$  are defined, point by point (meaning that for each point  $x^\lambda$  we consider a reference frame that is locally freely falling at  $x$ , with local coordinates  $X_x^\alpha$ ) by  $g_{\mu\nu}(x) = \eta_{\alpha\beta} \partial X_x^\alpha(x) / \partial x^\mu \partial X_x^\beta(x) / \partial x^\nu$ . Because of the nonlinearity of the functions  $X^\alpha(x)$ , the functions  $g_{\mu\nu}(x)$  generally depend in a nontrivial way on the coordinates  $x^\lambda$ .

The local chrono-geometry of space-time thus appears to be given, not by the simple Minkowskian metric (2), with constant coefficients  $\eta_{\mu\nu}$ , but by a quadratic metric of a much more general type, Equation (6), with coefficients  $g_{\mu\nu}(x)$  that vary from point to point. Such general metric spaces had been introduced and studied by Gauss and Riemann in the nineteenth century (in the case where the quadratic form (6) is positive definite). They carry the name *Riemannian spaces* or *curved spaces*. (In the case of interest for Einstein's theory, where the quadratic form (6) is not positive definite, one speaks of a pseudo-Riemannian metric.)

We do not have the space here to explain in detail the various geometric structures in a Riemannian space that are derivable from the data of the infinitesimal interval (6). Let us note simply that given Equation (6), which gives the distance  $ds$  between two infinitesimally separated points, we are able, through integration along a curve, to define the length of an arbitrary curve connecting two widely separated points  $A$  and  $B$ :  $L_{AB} = \int_A^B ds$ . One can then define the "straightest possible line" between two given points  $A$  and  $B$  to be the shortest line, in other words the curve that minimizes (or, more generally, extremizes) the integrated distance  $L_{AB}$ . These straightest possible lines are called *geodesic curves*. To give a simple example, the geodesics of a spherical surface (like the surface of the Earth) are the great circles (with radius equal to the radius of the sphere). If one mathematically writes the condition for a curve, as given by its parametric representation  $x^\mu = x^\mu(s)$ , where  $s$  is the length along the curve, to extremize the total length  $L_{AB}$  one finds that  $x^\mu(s)$  must satisfy the following second-order differential equation:

$$\frac{d^2 x^\lambda}{ds^2} + \Gamma_{\mu\nu}^\lambda(x) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0, \quad (7)$$

where the quantities  $\Gamma_{\mu\nu}^\lambda$ , known as the *Christoffel coefficients* or *connection coefficients*, are calculated, at each point  $x$ , from the *metric components*  $g_{\mu\nu}(x)$  by the equation

$$\Gamma_{\mu\nu}^\lambda \equiv \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}), \quad (8)$$

where  $g^{\mu\nu}$  denotes the matrix inverse to  $g_{\mu\nu}$  ( $g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu_\nu$  where the Kronecker symbol  $\delta^\mu_\nu$  is equal to 1 when  $\mu = \nu$  and 0 otherwise) and where  $\partial_\mu \equiv \partial/\partial x^\mu$  denotes the partial derivative with respect to the coordinate  $x^\mu$ . To give a very simple example: in the Poincaré-Minkowski space-time the components of the metric are constant,  $g_{\mu\nu} = \eta_{\mu\nu}$  (when we use an inertial reference frame). Because of this, the connection coefficients (8) vanish in an inertial reference frame, and the differential equation for geodesics reduces to  $d^2 x^\lambda/ds^2 = 0$ , whose solutions are ordinary straight lines:  $x^\lambda(s) = a^\lambda s + b^\lambda$ . On the other hand, in a general “curved” space-time (meaning one with components  $g_{\mu\nu}$  that depend in an arbitrary way on the point  $x$ ) the geodesics cannot be *globally* represented by straight lines. One can nevertheless show that it always remains possible, for any  $g_{\mu\nu}(x)$  whatsoever, to change coordinates  $x^\mu \rightarrow X^\alpha(x)$  in such a way that the connection coefficients  $\Gamma^\alpha_{\beta\gamma}$ , in the new system of coordinates  $X^\alpha$ , vanish *locally*, at a given point  $X_0^\alpha$  (or even along an arbitrary curve). Such *locally geodesic* coordinate systems realize Einstein’s equivalence principle mathematically: up to terms of second order, the components  $g_{\alpha\beta}(X)$  of a “curved” metric in locally geodesic coordinates  $X^\alpha$  ( $ds^2 = g_{\alpha\beta}(X) dX^\alpha dX^\beta$ ) can be identified with the components of a “flat” Poincaré-Minkowski metric:  $g_{\alpha\beta}(X) = \eta_{\alpha\beta} + \mathcal{O}((X - X_0)^2)$ , where  $X_0$  is the point around which we expand.

## 5 Einstein’s Equations: Elastic Space-Time

Having postulated that a consistent relativistic theory of the gravitational field should include the consideration of a far-reaching generalization of the Poincaré-Minkowski space-time, Equation (6), Einstein concluded that the same ten functions  $\mathbf{g}_{\mu\nu}(x)$  should describe both the **g**eometry of space-time as well as **g**ravitation. He therefore got down to the task of finding which equations must be satisfied by the “geometric-gravitational field”  $\mathbf{g}_{\mu\nu}(x)$ . He was guided in this search by three principles. The first was the *principle of general relativity*, which asserts that in the presence of a gravitational field one should be able to write the fundamental laws of physics (including those governing the gravitational field itself) in the same way in any coordinate system whatsoever. The second was that the “source” of the gravitational field should be the energy-momentum tensor  $T^{\mu\nu}$ . The third was a principle of *correspondence* with earlier physics: in the limit where one neglects gravitational effects,  $\mathbf{g}_{\mu\nu}(x) = \eta_{\mu\nu}$  should be a solution of the equations being sought, and there should also be a so-called *Newtonian* limit where the new theory reduces to Newton’s theory of gravity.

Note that the principle of general relativity (contrary to appearances and contrary to what Einstein believed for several years) has a different physical status than the principle of special relativity. The principle of special relativity was a symmetry principle for the structure of space-time that asserted that physics is *the same* in a particular class of reference frames, and therefore that certain “corresponding” phenomena occur in exactly the same way in different reference frames (“active” transformations). On the other hand, the principle of general relativity is a *principle of indifference*: the phenomena do not (in



general) take place in the same way in different coordinate systems. However, none of these (extended) coordinate systems enjoys any privileged status with respect to the others.

The principle asserting that the energy-momentum tensor  $T^{\mu\nu}$  should be the source of the gravitational field is founded on two ideas: the relations  $E = m_i c^2$  and the weak principle of equivalence  $m_i = m_g$  show that, in the Newtonian limit, the source of gravitation, the gravitational mass  $m_g$ , is equal to the total energy of the body considered, or in other words the integral over space of the energy density  $T^{00}$ , up to the factor  $c^{-2}$ . Therefore at least one of the components of the tensor  $T^{\mu\nu}$  must play the role of source for the gravitational field. However, since the gravitational field is encoded, according to Einstein, by the ten components of the metric  $g_{\mu\nu}$ , it is natural to suppose that the source for  $g_{\mu\nu}$  must also have ten components, which is precisely the case for the (symmetric) tensor  $T^{\mu\nu}$ .

In November of 1915, after many years of conceptually arduous work, Einstein wrote the final form of the theory of general relativity [6]. *Einstein's equations* are non-linear, second-order partial differential equations for the geometric-gravitational field  $g_{\mu\nu}$ , containing the energy-momentum tensor  $T_{\mu\nu} \equiv g_{\mu\kappa} g_{\nu\lambda} T^{\kappa\lambda}$  on the right-hand side. They are written as follows:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (9)$$

where  $G$  is the (Newtonian) gravitational constant,  $c$  is the speed of light, and  $R \equiv g^{\mu\nu} R_{\mu\nu}$  and the *Ricci tensor*  $R_{\mu\nu}$  are calculated as a function of the connection coefficients  $\Gamma_{\mu\nu}^\lambda$  (8) in the following way:

$$R_{\mu\nu} \equiv \partial_\alpha \Gamma_{\mu\nu}^\alpha - \partial_\nu \Gamma_{\mu\alpha}^\alpha + \Gamma_{\beta\alpha}^\alpha \Gamma_{\mu\nu}^\beta - \Gamma_{\beta\nu}^\alpha \Gamma_{\mu\alpha}^\beta. \quad (10)$$

One can show that, in a four-dimensional space-time, the three principles we have described previously uniquely determine Einstein's equations (9). It is nevertheless remarkable that these equations may also be developed from points of view that are completely different from the one taken by Einstein. For example, in the 1960s various authors (in particular Feynman, Weinberg and Deser; see references in [4]) showed that Einstein's equations could be obtained from a purely *dynamical* approach, founded on the consistency of interactions of a long-range spin 2 field, without making any appeal, as Einstein had, to the *geometric* notions coming from mathematical work on Riemannian spaces. Let us also note that if we relax one of the principles described previously (as Einstein did in 1917) we can find a generalization of Equation (9) in which one adds the term  $+\Lambda g_{\mu\nu}$  to the left-hand side, where  $\Lambda$  is the so-called *cosmological constant*. Such a modification was proposed by Einstein in 1917 in order to be able to write down a globally homogeneous and *stationary* cosmological solution. Einstein rejected this additional term after work by Friedmann (1922) showed the existence of *expanding* cosmological solutions of general relativity and after the observational discovery (by Hubble in 1929) of the expanding motion of galaxies within the universe. However, recent cosmological data have once again

made this possibility fashionable, although in the fundamental physics of today one tends to believe that a term of the type  $\Lambda g_{\mu\nu}$  should be considered as a particular physical contribution to the right-hand side of Einstein's equations (more precisely, as the stress-energy tensor of the *vacuum*,  $T_{\mu\nu}^V = -\frac{c^4}{8\pi G} \Lambda g_{\mu\nu}$ ), rather than as a universal geometric modification of the left-hand side.

Let us now comment on the physical meaning of Einstein's equations (9). The essential new idea is that the chrono-geometric structure of space-time, Equation (6), in other words the structure that underlies all of the measurements that one could locally make of duration,  $dT$ , and of distance,  $dL$ , (we recall that, locally,  $ds^2 = dL^2 - c^2 dT^2$ ) is no longer a rigid structure that is given a priori, once and for all (as was the case for the structure of Poincaré-Minkowski space-time), but instead has become a *field*, a dynamical or *elastic* structure, which is created and/or deformed by the presence of an energy-momentum distribution. See Figure 2, which visualizes the “elastic” geometry of space-time in the theory of general relativity by representing, around each point  $x$ , the locus of points (assumed to be infinitesimally close to  $x$ ) separated from  $x$  by a constant (squared) interval:  $ds^2 = \varepsilon^2$ . As in the case of Poincaré-Minkowski space-time (Figure 1), one arrives at a “field” of hyperboloids. However, this field of hyperboloids no longer has a “rigid” and homogeneous structure.

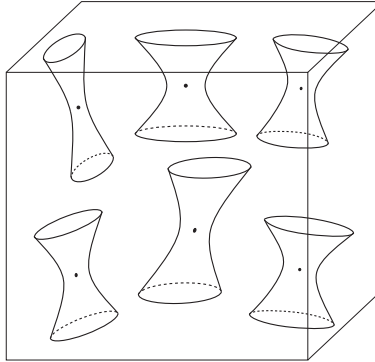


Figure 2: “Elastic” space-time geometry in the theory of general relativity. This geometry is visualized by representing, around each space-time point  $x$ , the locus of points separated from  $x$  by a given small positive (squared) interval.

The *space-time field*  $g_{\mu\nu}(x)$  describes the variation from point to point of the chrono-geometry as well as all gravitational effects. The simplest example of space-time chrono-geometric *elasticity* is the effect that the proximity of a mass has on the “local rate of flow for time.” In concrete terms, if you separate two twins at birth, with one staying on the surface of the Earth and the other going to live on the peak of a very tall mountain (in other words farther from the Earth’s center), and then reunite them after 100 years, the “highlander” will be older (will have lived longer) than the twin who stayed on the valley floor. Everything takes place as if time flows more slowly the closer one is to

a given distribution of mass-energy. In mathematical terms this effect is due to the fact that the coefficient  $g_{00}(x)$  of  $(dx^0)^2$  in Equation (6) is deformed with respect to its value in special relativity,  $g_{00}^{\text{Minkowski}} = \eta_{00} = -1$ , to become  $g_{00}^{\text{Einstein}}(x) \simeq -1 + 2GM/c^2 r$ , where  $M$  is the Earth's mass (in our example) and  $r$  the distance to the center of the Earth. In the example considered above of terrestrial twins the effect is extremely small (a difference in the amount of time lived of about one second over 100 years), but the effect is real and has been verified many times using atomic clocks (see the references in [4]). Let us mention that today this “Einstein effect” has important practical repercussions, for example in aerial or maritime navigation, for the piloting of automobiles, or even farm machinery, etc. In fact, the GPS (Global Positioning System), which uses the data transmitted by a constellation of atomic clocks on board satellites, incorporates the Einsteinian deformation of space-time chrono-geometry into its software. The effect is only on the order of one part in a billion, but if it were not taken into account, it would introduce an unacceptably large error into the GPS, which would continually grow over time. Indeed, GPS performance relies on the high stability of the orbiting atomic clocks, a stability better than  $10^{-13}$ , or in other words 10,000 times greater than the apparent change in frequency ( $\sim 10^{-9}$ ) due to the Einsteinian deformation of the chrono-geometry.

## 6 The Weak-Field Limit and the Newtonian Limit

To understand the physical consequences of Einstein's equations (9), it is useful to begin by considering the limiting case of *weak* geometric-gravitational fields, namely the case where  $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$ , with perturbations  $h_{\mu\nu}(x)$  that are very small with respect to unity:  $|h_{\mu\nu}(x)| \ll 1$ . In this case, a simple calculation (that we encourage the reader to perform) starting from Definitions (8) and (10) above, leads to the following explicit form of Einstein's equations (where we ignore terms of order  $h^2$  and  $hT$ ):

$$\square h_{\mu\nu} - \partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\alpha\mu} + \partial_{\mu\nu} h^\alpha_\alpha = -\frac{16\pi G}{c^4} \tilde{T}_{\mu\nu}, \quad (11)$$

where  $\square = \eta^{\mu\nu} \partial_{\mu\nu} = \Delta - \partial_0^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - c^{-2} \partial^2/\partial t^2$  denotes the “flat” d'Alembertian (or wave operator;  $x^\mu = (ct, x, y, z)$ ), and where indices in the upper position have been raised by the inverse  $\eta^{\mu\nu}$  of the flat metric  $\eta_{\mu\nu}$  (numerically  $\eta^{\mu\nu} = \eta_{\mu\nu}$ , meaning that  $-\eta^{00} = \eta^{11} = \eta^{22} = \eta^{33} = +1$ ). For example  $\partial^\alpha h_{\alpha\nu}$  denotes  $\eta^{\alpha\beta} \partial_\alpha h_{\beta\nu}$  and  $h^\alpha_\alpha \equiv \eta^{\alpha\beta} h_{\alpha\beta} = -h_{00} + h_{11} + h_{22} + h_{33}$ . The “source”  $\tilde{T}_{\mu\nu}$  appearing on the right-hand side of (11) denotes the combination  $\tilde{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} T^\alpha_\alpha \eta_{\mu\nu}$  (when space-time is four-dimensional).

The “linearized” approximation (11) of Einstein's equations is analogous to Maxwell's equations

$$\square A_\mu - \partial_\mu \partial^\alpha A_\alpha = -4\pi J_\mu, \quad (12)$$

connecting the electromagnetic four-potential  $A_\mu \equiv \eta_{\mu\nu} A^\nu$  (where  $A^0 = V$ ,  $A^i = \mathbf{A}$ ,  $i = 1, 2, 3$ ) to the four-current density  $J_\mu \equiv \eta_{\mu\nu} J^\nu$  (where  $J^0 = \rho$  is the

charge density and  $J^i = \mathbf{J}$  is the current density). Another analogy is that the structure of the left-hand side of Maxwell's equations implies that the "source"  $J_\mu$  appearing on the right-hand side must satisfy  $\partial^\mu J_\mu = 0$  ( $\partial^\mu \equiv \eta^{\mu\nu} \partial_\nu$ ), which expresses the conservation of electric charge. Likewise, the structure of the left-hand side of the linearized form of Einstein's equations (11) implies that the "source"  $T_{\mu\nu} = \tilde{T}_{\mu\nu} - \frac{1}{2} \tilde{T}^\alpha_\alpha \eta_{\mu\nu}$  must satisfy  $\partial^\mu T_{\mu\nu} = 0$ , which expresses the conservation of energy and momentum of matter. (The structure of the left-hand side of the exact form of Einstein's equations (9) implies that the source  $T_{\mu\nu}$  must satisfy the more complicated equation  $\partial_\mu T^{\mu\nu} + \Gamma^\mu_{\sigma\mu} T^{\sigma\nu} + \Gamma^\nu_{\sigma\mu} T^{\mu\sigma} = 0$ , where the terms in  $\Gamma T$  can be interpreted as describing an exchange of energy and momentum between matter and the gravitational field.) The major difference is that, in the case of electromagnetism, the field  $A_\mu$  and its source  $J_\mu$  have a single space-time index, while in the gravitational case the field  $h_{\mu\nu}$  and its source  $\tilde{T}_{\mu\nu}$  have two space-time indices. We shall return later to this analogy/difference between  $A_\mu$  and  $h_{\mu\nu}$  which suggests the existence of a certain relation between gravitation and electromagnetism.

We recover the Newtonian theory of gravitation as the limiting case of Einstein's theory by assuming not only that the gravitational field is a weak deformation of the flat Minkowski space-time ( $h_{\mu\nu} \ll 1$ ), but also that the field  $h_{\mu\nu}$  is slowly varying ( $\partial_0 h_{\mu\nu} \ll \partial_i h_{\mu\nu}$ ) and that its source  $T_{\mu\nu}$  is non-relativistic ( $T_{ij} \ll T_{0i} \ll T_{00}$ ). Under these conditions Equation (11) leads to a Poisson-type equation for the purely temporal component,  $h_{00}$ , of the space-time field,

$$\Delta h_{00} = -\frac{16\pi G}{c^4} \tilde{T}_{00} = -\frac{8\pi G}{c^4} (T_{00} + T_{ii}) \simeq -\frac{8\pi G}{c^4} T_{00}, \quad (13)$$

where  $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$  is the Laplacian. Recall that, according to Laplace and Poisson, Newton's theory of gravity is summarized by saying that the gravitational field is described by a single potential  $U(x)$ , produced by the mass density  $\rho(x)$  according to the Poisson equation  $\Delta U = -4\pi G\rho$ , that determines the acceleration of a test particle placed in the exterior field  $U(x)$  according to the equation  $d^2 x^i/dt^2 = \partial_i U(x) \equiv \partial U/\partial x^i$ . Because of the relation  $m_i = m_g = E/c^2$  one can identify  $\rho = T^{00}/c^2$ . We therefore find that (13) reproduces the Poisson equation if  $h_{00} = +2U/c^2$ . It therefore remains to verify that Einstein's theory indeed predicts that a non-relativistic test particle is accelerated by a space-time field according to  $d^2 x^i/dt^2 \simeq \frac{1}{2} c^2 \partial_i h_{00}$ . Einstein understood that this was a consequence of the equivalence principle. In fact, as we discussed in Section 4 above, the principle of equivalence states that the gravitational field is (locally) erased in a locally inertial reference frame  $X^\alpha$  (such that  $g_{\alpha\beta}(X) = \eta_{\alpha\beta} + \mathcal{O}((X - X_0)^2)$ ). In such a reference frame, the laws of special relativity apply at the point  $X_0$ . In particular an isolated (and electrically neutral) body must satisfy a principle of inertia in this frame: its center of mass moves in a straight line at constant speed. In other words it satisfies the equation of motion  $d^2 X^\alpha/ds^2 = 0$ . By passing back to an arbitrary (extended) coordinate system  $x^\mu$ , one verifies that this equation for inertial motion transforms into the geodesic equation (7). Therefore (7) describes falling bodies, such as they are observed in arbitrary extended reference frames (for example a

reference frame at rest with respect to the Earth or at rest with respect to the center of mass of the solar system). From this one concludes that the relativistic analog of the Newtonian field of gravitational acceleration,  $\mathbf{g}(x) = \nabla U(x)$ , is  $g^\lambda(x) \equiv -c^2 \Gamma_{\mu\nu}^\lambda dx^\mu/ds dx^\nu/ds$ . By considering a particle whose motion is slow with respect to the speed of light ( $dx^i/ds \ll dx^0/ds \simeq 1$ ) one can easily verify that  $g^i(x) \simeq -c^2 \Gamma_{00}^i$ . Finally, by using the definition (8) of  $\Gamma_{\mu\nu}^\alpha$ , and the hypothesis of weak fields, one indeed verifies that  $g^i(x) \simeq \frac{1}{2} c^2 \partial_i h_{00}$ , in perfect agreement with the identification  $h_{00} = 2U/c^2$  anticipated above. We encourage the reader to personally verify this result, which contains the very essence of Einstein's theory: gravitational motion is no longer described as being due to a force, but is identified with motion that is "as inertial as possible" within a space-time whose chrono-geometry is deformed in the presence of a mass-energy distribution.

Finding the Newtonian theory as a limiting case of Einstein's theory is obviously a necessity for seriously considering this new theory. But of course, from the very beginning Einstein explored the observational consequences of general relativity that go beyond the Newtonian description of gravitation. We have already mentioned one of these above: the fact that  $g_{00} = \eta_{00} + h_{00} \simeq -1 + 2U(x)/c^2$  implies a distortion in the relative measurement of time in the neighborhood of massive bodies. In 1907 (as soon as he had developed the principle of equivalence, and long before he had obtained the field equations of general relativity) Einstein had predicted the existence of such a distortion for measurements of time and frequency in the presence of an external gravitational field. He realized that this should have observable consequences for the frequency, as observed on Earth, of the spectral rays emitted from the surface of the Sun. Specifically, a spectral ray of (proper local) frequency  $\nu_0$  emitted from a point  $x_0$  where the (stationary) gravitational potential takes the value  $U(\mathbf{x}_0)$  and observed (via electromagnetic signals) at a point  $x$  where the potential is  $U(\mathbf{x})$  should appear to have a frequency  $\nu$  such that

$$\frac{\nu}{\nu_0} = \sqrt{\frac{g_{00}(x_0)}{g_{00}(x)}} \simeq 1 + \frac{1}{c^2} [U(\mathbf{x}) - U(\mathbf{x}_0)]. \quad (14)$$

In the case where the point of emission  $x_0$  is in a gravitational potential well deeper than the point of observation  $x$  (meaning that  $U(\mathbf{x}_0) > U(\mathbf{x})$ ) one has  $\nu < \nu_0$ , in other words a *reddening* effect on frequencies. This effect, which was predicted by Einstein in 1907, was unambiguously verified only in the 1960s, in experiments by Pound and collaborators over a height of about twenty meters. The most precise verification (at the level of  $\sim 10^{-4}$ ) is due to Vessot and collaborators, who compared a hydrogen maser, launched aboard a rocket that reached about 10,000 km in altitude, to a clock of similar construction on the ground. Other experiments compared the times shown on clocks placed aboard airplanes to clocks remaining on the ground. (For references to these experiments see [4].) As we have already mentioned, the "Einstein effect" (14) must be incorporated in a crucial way into the software of satellite positioning systems such as the GPS.

In 1907, Einstein also pointed out that the equivalence principle would suggest that light rays should be deflected by a gravitational field. Indeed, a generalization of the reasoning given above for the motion of particles in an external gravitational field, based on the principle of equivalence, shows that light must itself follow a trajectory that is “as inertial as possible,” meaning a geodesic of the curved space-time. Light rays must therefore satisfy the geodesic equation (7). (The only difference from the geodesics followed by material particles is that the parameter  $s$  in Equation (7) can no longer be taken equal to the “length” along the geodesic, since a “light” geodesic must also satisfy the constraint  $g_{\mu\nu}(x) dx^\mu dx^\nu = 0$ , ensuring that its speed is equal to  $c$ , when it is measured in a locally inertial reference frame.) Starting from Equation (7) one can therefore calculate to what extent light is deflected when it passes through the neighborhood of a large mass (such as the Sun). One nevertheless soon realizes that in order to perform this calculation one must know more than the component  $h_{00}$  of the gravitational field. The other components of  $h_{\mu\nu}$ , and in particular the spatial components  $h_{ij}$ , come into play in a crucial way in this calculation. This is why it was only in November of 1915, after having obtained the (essentially) final form of his theory, that Einstein could predict the total value of the deflection of light by the Sun. Starting from the linearized form of Einstein’s equations (11) and continuing by making the “non-relativistic” simplifications indicated above ( $T_{ij} \ll T_{0i} \ll T_{00}$ ,  $\partial_0 h \ll \partial_i h$ ) it is easy to see that the spatial component  $h_{ij}$ , like  $h_{00}$ , can be written (after a helpful choice of coordinates) in terms of the Newtonian potential  $U$  as  $h_{ij}(x) \simeq +2U(x)\delta_{ij}/c^2$ , where  $\delta_{ij}$  takes the value 1 if  $i = j$  and 0 otherwise ( $i, j = 1, 2, 3$ ). By inserting this result, as well as the preceding result  $h_{00} = +2U/c^2$ , into the geodesic equation (7) for the motion of light, one finds (as Einstein did in 1915) that general relativity predicts that the Sun should deflect a ray of light by an angle  $\theta = 4GM/(c^2b)$  where  $b$  is the impact parameter of the ray (meaning its minimum distance from the Sun). As is well known, the confirmation of this effect in 1919 (with rather weak precision) made the theory of general relativity and its creator famous.

## 7 The Post-Newtonian Approximation and Experimental Confirmations in the Regime of Weak and Quasi-Stationary Gravitational Fields

We have already pointed out some of the experimental confirmations of the theory of general relativity. At present, the extreme precision of certain measurements of time or frequency in the solar system necessitates a very careful account of the modifications brought by general relativity to the Newtonian description of space-time. As a consequence, general relativity is used in a great number of situations, from astronomical or geophysical research (such as very long range radio interferometry, radar tracking of the planets, and laser tracking of the Moon or artificial satellites) to metrological, geodesic or other applica-

tions (such as the definition of international atomic time, precision cartography, and the G.P.S.). To do this, the so-called *post-Newtonian* approximation has been developed. This method involves working in the Newtonian limit sketched above while keeping the terms of higher order in the small parameter

$$\varepsilon \sim \frac{v^2}{c^2} \sim |h_{\mu\nu}| \sim |\partial_0 h / \partial_i h|^2 \sim |T^{0i}/T^{00}|^2 \sim |T^{ij}/T^{00}|,$$

where  $v$  denotes a characteristic speed for the elements in the system considered.

For all present applications of general relativity to the solar system it suffices to include the *first post-Newtonian approximation*, in other words to keep the relative corrections of order  $\varepsilon$  to the Newtonian predictions. Since the theory of general relativity was poorly verified for a long time, one found it useful (as in the pioneering work of A. Eddington, generalized in the 1960s by K. Nordtvedt and C.M. Will) to study not only the precise predictions of the equations (9) defining Einstein's theory, but to also consider possible deviations from these predictions. These possible deviations were parameterized by means of several non-dimensional "post-Newtonian" parameters. Among these parameters, two play a key role:  $\gamma$  and  $\beta$ . The parameter  $\gamma$  describes a possible deviation from general relativity that comes into play starting at the linearized level, in other words one that modifies the linearized approximation given above. More precisely, it is defined by writing that the difference  $h_{ij} \equiv g_{ij} - \delta_{ij}$  between the spatial metric and the Euclidean metric can take the value  $h_{ij} = 2\gamma U \delta_{ij}/c^2$  (in a suitable coordinate system), rather than the value  $h_{ij}^{\text{GR}} = 2U \delta_{ij}/c^2$  that it takes in general relativity, thus differing by a factor  $\gamma$ . Therefore, by definition  $\gamma$  takes the value 1 in general relativity, and  $\gamma - 1$  measures the possible deviation with respect to this theory. As for the parameter  $\beta$  (or rather  $\beta - 1$ ), it measures a possible deviation (with respect to general relativity) in the value of  $h_{00} \equiv g_{00} - \eta_{00}$ . The value of  $h_{00}$  in general relativity is  $h_{00}^{\text{GR}} = 2U/c^2 - 2U^2/c^4$ , where the first term (discussed above) reproduces the Newtonian approximation (and cannot therefore be modified, as the idea is to parameterize gravitational physics beyond Newtonian predictions) and where the second term is obtained by solving Einstein's equations (9) at the second order of approximation. One then writes an  $h_{00}$  of a more general parameterized type,  $h_{00} = 2U/c^2 - 2\beta U^2/c^4$ , where, by definition,  $\beta$  takes the value 1 in general relativity. Let us finally point out that the parameters  $\gamma - 1$  and  $\beta - 1$  completely parameterize the post-Newtonian regime of the simplest theoretical alternatives to general relativity, namely the tensor-scalar theories of gravitation. In these theories, the gravitational interaction is carried by two fields at the same time: a massless tensor (spin 2) field coupled to  $T^{\mu\nu}$ , and a massless scalar (spin 0) field  $\varphi$  coupled to the trace  $T^\alpha_\alpha$ . In this case the parameter  $-(\gamma - 1)$  plays the key role of measuring the ratio between the scalar coupling and the tensor coupling.

All of the experiments performed to date within the solar system are compatible with the predictions of general relativity. When they are interpreted in terms of the post-Newtonian (and "post-Einsteinian") parameters  $\gamma - 1$  and  $\beta - 1$ , they lead to strong constraints on possible deviations from Einstein's theory. We make note of the following among tests performed in the solar system:

the deflection of electromagnetic waves in the neighborhood of the Sun, the gravitational delay (‘Shapiro effect’) of radar signals bounced from the Viking lander on Mars, the global analysis of solar system dynamics (including the advance of planetary perihelia), the sub-centimeter measurement of the Earth-Moon distance obtained from laser signals bounced off of reflectors on the Moon’s surface, etc. At present (October of 2006) the most precise test (that has been published) of general relativity was obtained in 2003 by measuring the ratio  $1 + y \equiv f/f_0$  between the frequency  $f_0$  of radio waves sent from Earth to the Cassini space probe and the frequency  $f$  of coherent radio waves sent back (with the same local frequency) from Cassini to Earth and compared (on Earth) to the emitted frequency  $f_0$ . The main contribution to the small quantity  $y$  is an effect equal, in general relativity, to  $y_{GR} = 8(GM/c^3 b) db/dt$  (where  $b$  is, as before, the impact parameter) due to the propagation of radio waves in the geometry of a space-time deformed by the Sun:  $ds^2 \simeq -(1 - 2U/c^2) c^2 dt^2 + (1 + 2U/c^2)(dx^2 + dy^2 + dz^2)$ , where  $U = GM/r$ . The maximum value of the frequency change predicted by general relativity was only  $|y_{GR}| \lesssim 2 \times 10^{-10}$  for the best observations, but thanks to an excellent frequency stability  $\sim 10^{-14}$  (after correction for the perturbations caused by the solar corona) and to a relatively large number of individual measurements spread over 18 days, this experiment was able to verify Einstein’s theory at the remarkable level of  $\sim 10^{-5}$  [7]. More precisely, when this experiment is interpreted in terms of the post-Newtonian parameters  $\gamma - 1$  and  $\beta - 1$ , it gives the following limit for the parameter  $\gamma - 1$  [7]

$$\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}. \quad (15)$$

As for the best present-day limit on the parameter  $\beta - 1$ , it is smaller than  $10^{-3}$  and comes from the non-observation, in the data from lasers bounced off of the Moon, of any eventual polarization of the Moon’s orbit in the direction of the Sun (‘Nordtvedt effect’; see [4] for references)

$$4(\beta - 1) - (\gamma - 1) = -0.0007 \pm 0.0010. \quad (16)$$

Although the theory of general relativity is one of the best verified theories in physics, scientists continue to design and plan new or increasingly precise tests of the theory. This is the case in particular for the space mission Gravity Probe B (launched by NASA in April of 2004) whose principal aim is to directly observe a prediction of general relativity that states (intuitively speaking) that space is not only “elastic,” but also “fluid.” In the nineteenth century Foucault invented both the gyroscope and his famous pendulum in order to render Newton’s absolute (and rigid) space directly observable. His experiments in fact showed that, for example, a gyroscope on the surface of the Earth continued, despite the Earth’s rotation, to align itself in a direction that is “fixed” with respect to the distant stars. However, in 1918, when Lense and Thirring analyzed some of the consequences of the (linearized) Einstein equations (11), they found that general relativity predicts, among other things, the following phenomenon: the rotation of the Earth (or any other ball of matter) creates a particular deformation of the chrono-geometry of space-time. This deformation is described by the “gravito-magnetic” components  $h_{0i}$  of the metric, and induces an effect analogous to



the “rotation drag” effect caused by a ball of matter turning in a fluid: the rotation of the Earth (minimally) drags all of the space around it, causing it to continually “turn,” as a fluid would.<sup>3</sup> This “rotation of space” translates, in an observable way, into a violation of the effects predicted by Newton and confirmed by Foucault’s experiments: in particular, a gyroscope no longer aligns itself in a direction that is “fixed in absolute space,” rather its axis of rotation is “dragged” by the rotating motion of the local space where it is located. This effect is much too small to be visible in Foucault’s experiments. Its observation by Gravity Probe B (see [8] and the contribution of John Mester to this Poincaré seminar) is important for making Einstein’s revolutionary notion of a fluid space-time tangible to the general public.

Up till now we have only discussed the regime of weak and slowly varying gravitational fields. The theory of general relativity predicts the appearance of new phenomena when the gravitational field becomes strong and/or rapidly varying. (We shall not here discuss the cosmological aspects of relativistic gravitation.)

## 8 Strong Gravitational Fields and Black Holes

The regime of strong gravitational fields is encountered in the physics of *gravitationally condensed bodies*. This term designates the final states of stellar evolution, and in particular neutron stars and black holes. Recall that most of the life of a star is spent slowly burning its nuclear fuel. This process causes the star to be structured as a series of layers of differentiated nuclear structure, surrounding a progressively denser core (an “onion-like” structure). When the initial mass of the star is sufficiently large, this process ends into a catastrophic phenomenon: the core, already much denser than ordinary matter, collapses in on itself under the influence of its gravitational self-attraction. (This implosion of the central part of the star is, in many cases, accompanied by an explosion of the outer layers of the star—a supernova.) Depending on the quantity of mass that collapses with the core of a star, this collapse can give rise to either a neutron star or a black hole.

A *neutron star* condenses a mass on the order of the mass of the Sun inside a radius on the order of 10 km. The density in the interior of a neutron star (named thus because neutrons dominate its nuclear composition) is more than 100 million tons per cubic centimeter ( $10^{14}$  g/cm<sup>3</sup>)! It is about the same as the density in the interior of atomic nuclei. What is important for our discussion is that the deformation away from the Minkowski metric in the immediate neighborhood of a neutron star, measured by  $h_{00} \sim h_{ii} \sim 2GM/c^2 R$ , where  $R$  is the radius of the star, is no longer a small quantity, as it was in the solar system. In fact, while  $h \sim 2GM/c^2 R$  is on the order of  $10^{-9}$  for the Earth and  $10^{-6}$  for the Sun, one finds that  $h \sim 0.4$  for a typical neutron star ( $M \simeq 1.4 M_\odot$ ,  $R \sim 10$

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<sup>3</sup>Recent historical work (by Herbert Pfister) has in fact shown that this effect had already been derived by Einstein within the framework of the provisory relativistic theory of gravity that he started to develop in 1912 in collaboration with Marcel Grossmann.

km). One thus concludes that it is no longer possible, as was the case in the solar system, to study the structure and physics of neutron stars by using the post-Newtonian approximation outlined above. One must consider the exact form of Einstein's equations (9), with all of their non-linear structure. Because of this, we expect that observations concerning neutron stars will allow us to confirm (or refute) the theory of general relativity in its strongly non-linear regime. We shall discuss such tests below in relation to observations of binary pulsars.

A *black hole* is the result of a *continued* collapse, meaning that it does not stop with the formation of an ultra-dense star (such as a neutron star). (The physical concept of a black hole was introduced by J.R. Oppenheimer and H. Snyder in 1939. The global geometric structure of black holes was not understood until some years later, thanks notably to the work of R. Penrose. For a historical review of the idea of black holes see [9].) It is a particular structure of curved space-time characterized by the existence of a boundary (called the “black hole surface” or “horizon”) between an exterior region, from which it is possible to emit signals to infinity, and an interior region (of space-time), within which any emitted signal remains trapped. See Figure 3.

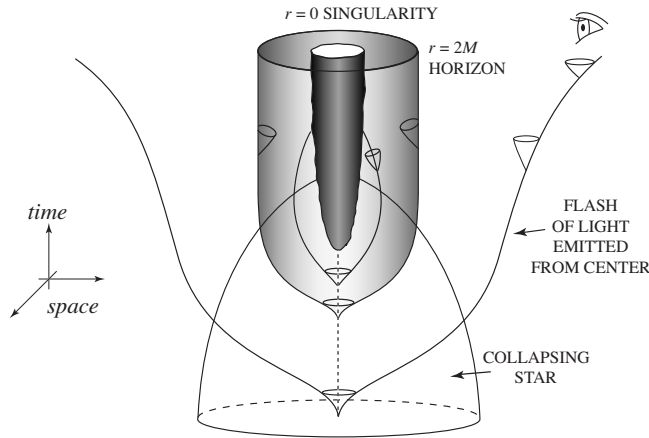


Figure 3: Schematic representation of the space-time for a black hole created from the collapse of a spherical star. Each cone represents the space-time history of a flash of light emitted from a point at a particular instant. (Such a “cone field” is obtained by taking the limit  $\varepsilon^2 = 0$  from Figure 2, and keeping only the upper part, in other words the part directed towards the future, of the double cones obtained as limits of the hyperboloids of Figure 2.) The interior of the black hole is shaded, its outer boundary being the “black hole surface” or “horizon.” The “inner boundary” (shown in dark grey) of the interior region of the black hole is a space-time singularity of the big-crunch type.

The cones shown in this figure are called “light cones.” They are defined as the locus of points (infinitesimally close to  $x$ ) such that  $ds^2 = 0$ , with  $dx^0 =$

$cdt \geq 0$ . Each represents the beginning of the space-time history of a flash of light emitted from a certain point in space-time. The cones whose vertices are located outside of the horizon (the shaded zone) will evolve by spreading out to infinity, thus representing the possibility for electromagnetic signals to reach infinity.

On the other hand, the cones whose vertices are located inside the horizon (the grey zone) will evolve without ever succeeding in escaping the grey zone. It is therefore impossible to emit an electromagnetic signal that reaches infinity from the grey zone. The horizon, namely the boundary between the shaded zone and the unshaded zone, is itself the history of a particular flash of light, emitted from the center of the star over the course of its collapse, such that it asymptotically stabilizes as a space-time cylinder. This space-time cylinder (the asymptotic horizon) therefore represents the space-time history of a bubble of light that, viewed locally, moves outward at the speed  $c$ , but which globally “runs in place.” This remarkable behavior is a striking illustration of the “fluid” character of space-time in Einstein’s theory. Indeed, one can compare the preceding situation with what may take place around the open drain of an emptying sink: a wave may move along the water, away from the hole, all the while running in place with respect to the sink because of the falling motion of the water in the direction of the drain.

Note that the temporal development of the interior region is limited, terminating in a *singularity* (the dark gray surface) where the curvature becomes infinite and where the classical description of space and time loses its meaning. This singularity is locally similar to the temporal inverse of a cosmological singularity of the big bang type. This is called a *big crunch*. It is a space-time frontier, beyond which space-time ceases to exist. The appearance of singularities associated with regions of strong gravitational fields is a generic phenomenon in general relativity, as shown by theorems of R. Penrose and S.W. Hawking.

Black holes have some remarkable properties. First, a *uniqueness* theorem (due to W. Israel, B. Carter, D.C. Robinson, G. Bunting, and P.O. Mazur) asserts that an isolated, stationary black hole (in Einstein-Maxwell theory) is completely described by three parameters: its mass  $M$ , its angular momentum  $J$ , and its electric charge  $Q$ . The exact solution (called the Kerr-Newman solution) of Einstein’s equations (11) describing a black hole with parameters  $M, J, Q$  is explicitly known. We shall here content ourselves with writing the space-time geometry in the simplest case of a black hole: the one in which  $J = Q = 0$  and the black hole is described only by its mass (a solution discovered by K. Schwarzschild in January of 1916):

$$ds^2 = - \left( 1 - \frac{2GM}{c^2 r} \right) c^2 dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (17)$$

We see that the purely temporal component of the metric,  $g_{00} = -(1 - 2GM/c^2 r)$ , vanishes when the radial coordinate  $r$  takes the value  $r = r_H \equiv 2GM/c^2$ . According to the earlier equation (14), it would therefore seem that light emitted from an arbitrary point on the sphere  $r_0 = r_H$ , when it is viewed by an observer

located anywhere in the exterior (in  $r > r_H$ ), would experience an infinite red-dening of its emission frequency ( $\nu/\nu_0 = 0$ ). In fact, the sphere  $r_H = 2GM/c^2$  is the *horizon* of the Schwarzschild black hole, and no particle (that is capable of emitting light) can remain at rest when  $r = r_H$  (nor, a fortiori, when  $r < r_H$ ). To study what happens at the horizon ( $r = r_H$ ) or in the interior ( $r < r_H$ ) of a Schwarzschild black hole, one must use other space-time coordinates than the coordinates  $(t, r, \theta, \varphi)$  used in Equation (17). The “big crunch” singularity in the interior of a Schwarzschild black hole, in the coordinates of (17), is located at  $r = 0$  (which does not describe, as one might believe, a point in space, but rather an instant in time).

The space-time metric of a black hole space-time, such as Equation (17) in the simple case  $J = Q = 0$ , allows one to study the influence of a black hole on particles and fields in its neighborhood. One finds that a black hole is a gravitational potential well that is so deep that any particle or wave that penetrates the interior of the black hole (the region  $r < r_H$ ) will never be able to come out again, and that the total energy of the particle or wave that “falls” into the black hole ends up augmenting the total mass-energy  $M$  of the black hole. By studying such black hole “accretion” processes with falling particles (following R. Penrose), D. Christodoulou and R. Ruffini showed that a black hole is not only a potential well, but also a physical object possessing a significant *free energy* that it is possible, in principle, to extract. Such *black hole energetics* is encapsulated in the “mass formula” of Christodoulou and Ruffini (in units where  $c = 1$ )

$$M^2 = \left( M_{\text{irr}} + \frac{Q^2}{4GM_{\text{irr}}} \right)^2 + \frac{J^2}{4G^2M_{\text{irr}}^2}, \quad (18)$$

where  $M_{\text{irr}}$  denotes the *irreducible mass* of the black hole, a quantity *that can only grow, irreversibly*. One deduces from (18) that a rotating ( $J \neq 0$ ) and/or charged ( $Q \neq 0$ ) black hole possesses a free energy  $M - M_{\text{irr}} > 0$  that can, in principle, be extracted through processes that reduce its angular momentum and/or its electric charge. Such black hole energy-extraction processes may lie at the origin of certain ultra-energetic astrophysical phenomena (such as quasars or gamma ray bursts). Let us note that, according to Equation (18), (rotating or charged) black holes are the largest reservoirs of free energy in the Universe: in fact, 29% of their mass energy can be stored in the form of rotational energy, and up to 50% can be stored in the form of electric energy. These percentages are much higher than the few percent of nuclear binding energy that is at the origin of all the light emitted by stars over their lifetimes. Even though there is not, at present, irrefutable proof of the existence of black holes in the universe, an entire range of very strong presumptive evidence lends credence to their existence. In particular, more than a dozen X-ray emitting binary systems in our galaxy are most likely made up of a black hole and an ordinary star. Moreover, the center of our galaxy seems to contain a very compact concentration of mass  $\sim 3 \times 10^6 M_\odot$  that is probably a black hole. (For a review of the observational data leading to these conclusions see, for example, Section 7.6 of the recent book by N. Straumann [6].)

The fact that a quantity associated with a black hole, here the irreducible mass  $M_{\text{irr}}$ , or, according to a more general result due to S.W. Hawking, the total area  $A$  of the surface of a black hole ( $A = 16 \pi G^2 M_{\text{irr}}^2$ ), can evolve only by irreversibly growing is reminiscent of the second law of thermodynamics. This result led J.D. Bekenstein to interpret the horizon area,  $A$ , as being proportional to the *entropy* of the black hole. Such a thermodynamic interpretation is reinforced by the study of the growth of  $A$  under the influence of external perturbations, a growth that one can in fact attribute to some local dissipative properties of the black hole surface, notably a surface viscosity and an electrical resistivity equal to 377 ohm (as shown in work by T. Damour and R.L. Znajek). These “thermodynamic” interpretations of black hole properties are based on simple analogies at the level of classical physics, but a remarkable result by Hawking showed that they have real content at the level of quantum physics. In 1974, Hawking discovered that the presence of a horizon in a black hole space-time affected the definition of a quantum particle, and caused a black hole to continuously emit a flux of particles having the characteristic spectrum (Planck spectrum) of thermal emission at the temperature  $T = 4 \hbar G \partial M / \partial A$ , where  $\hbar$  is the reduced Planck constant. By using the general thermodynamic relation connecting the temperature to the energy  $E = M$  and the entropy  $S$ ,  $T = \partial M / \partial S$ , we see from Hawking’s result (in conformity with Bekenstein’s ideas) that a black hole possesses an *entropy*  $S$  equal (again with  $c = 1$ ) to

$$S = \frac{1}{4} \frac{A}{\hbar G}. \quad (19)$$

The Bekenstein-Hawking formula (19) suggests an unexpected, and perhaps profound, connection between gravitation, thermodynamics, and quantum theory. See Section 11 below.

## 9 Binary Pulsars and Experimental Confirmations in the Regime of Strong and Radiating Gravitational Fields

*Binary pulsars* are binary systems made up of a pulsar (a rapidly spinning neutron star) and a very dense companion star (either a neutron star or a white dwarf). The first system of this type (called PSR B1913+16) was discovered by R.A. Hulse and J.H. Taylor in 1974 [10]. Today, a dozen are known. Some of these (including the first-discovered PSR B1913+16) have revealed themselves to be remarkable probes of relativistic gravitation and, in particular, of the regime of strong and/or radiating gravitational fields. The reason for which a binary pulsar allows for the probing of strong gravitational fields is that, as we have already indicated above, the deformation of the space-time geometry in the neighborhood of a neutron star is no longer a small quantity, as it is in the solar system. Rather, it is on the order of unity:  $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu} \sim 2GM/c^2 R \sim 0.4$ . (We note that this value is only 2.5 times smaller than in the extreme case of a

black hole, for which  $2GM/c^2R = 1$ .) Moreover, the fact that the gravitational interaction propagates at the speed of light (as indicated by the presence of the wave operator,  $\square = \Delta - c^{-2}\partial^2/\partial t^2$  in (11)) between the pulsar and its companion is found to play an observationally significant role for certain binary pulsars.

Let us outline how the observational data from binary pulsars are used to probe the regime of strong ( $h_{\mu\nu}$  on the order of unity) and/or radiative (effects propagating at the speed  $c$ ) gravitational fields. (For more details on the observational data from binary pulsars and their use in probing relativistic gravitation, see Michael Kramer's contribution to this Poincaré seminar.) Essentially, a pulsar plays the role of an extremely stable *clock*. Indeed, the “pulsar phenomenon” is due to the rotation of a bundle of electromagnetic waves, created in the neighborhood of the two magnetic poles of a strongly magnetized neutron star (with a magnetic field on the order of  $10^{12}$  Gauss,  $10^{12}$  times the size of the terrestrial magnetic field). Since the magnetic axis of a pulsar is not aligned with its axis of rotation, the rapid rotation of the pulsar causes the (inner) magnetosphere as a whole to rotate, and likewise the bundle of electromagnetic waves created near the magnetic poles. The pulsar is therefore analogous to a lighthouse that sweeps out space with two bundles (one per pole) of electromagnetic waves. Just as for a lighthouse, one does not see the pulsar from Earth except when the bundle sweeps the Earth, thus causing a flash of electromagnetic noise with each turn of the pulsar around itself (in some cases, one even sees a secondary flash, due to emission from the second pole, after each half-turn). One can then measure the time of arrival at Earth of (the center of) each flash of electromagnetic noise. The basic observational data of a pulsar are thus made up of a regular, discrete sequence of the *arrival times* at Earth of these flashes or “pulses.” This sequence is analogous to the signal from a clock: tick, tick, tick, ... Observationally, one finds that some pulsars (and in particular those that belong to binary systems) thus define clocks of a stability comparable to the best atomic clocks [11]. In the case of a solitary pulsar, the sequence of its arrival times is (in essence) a regular “arithmetic sequence,”  $T_N = aN + b$ , where  $N$  is an integer labelling the pulse considered, and where  $a$  is equal to the period of rotation of the pulsar around itself. In the case of a binary pulsar, the sequence of arrival times is a much richer signal, say  $T_N = aN + b + \Delta_N$ , where  $\Delta_N$  measures the deviation with respect to a regular arithmetic sequence. This deviation (after the subtraction of effects not connected to the orbital period of the pulsar) is due to a whole ensemble of physical effects connected to the orbital motion of the pulsar around its companion or, more precisely, around the center of mass of the binary system. Some of these effects could be predicted by a purely *Keplerian* description of the motion of the pulsar in space, and are analogous to the “Rømer effect” that allowed Rømer to determine, for the first time, the speed of light from the arrival times at Earth of light signals coming from Jupiter's satellites (the light signals coming from a body moving in orbit are “delayed” by the time taken by light to cross this orbit and arrive at Earth). Other effects can only be predicted and calculated by using a *relativistic* description, either of the orbital motion of the pulsar, or of the prop-

agation of electromagnetic signals between the pulsar and Earth. For example, the following facts must be accounted for: (i) the “pulsar clock” moves at a large speed (on the order of  $300 \text{ km/s} \sim 10^{-3}c$ ) and is embedded in the varying gravitational potential of the companion; (ii) the orbit of the pulsar is not a simple Keplerian ellipse, but (in general relativity) a more complicated orbit that traces out a “rosette” around the center of mass; (iii) the propagation of electromagnetic signals between the pulsar and Earth takes place in a space-time that is curved by both the pulsar and its companion, which leads to particular effects of relativistic delay; etc. Taking relativistic effects in the theoretical description of arrival times for signals emitted by binary pulsars into account thus leads one to write what is called a *timing formula*. This timing formula (due to T. Damour and N. Deruelle) in essence allows one to parameterize the sequence of arrival times,  $T_N = aN + b + \Delta_N$ , in other words to parameterize  $\Delta_N$ , as a function of a set of “phenomenological parameters” that include not only the so-called “Keplerian” parameters (such as the orbital period  $P$ , the projection of the semi-major axis of the pulsar’s orbit along the line of sight  $x_A = a_A \sin i$ , and the eccentricity  $e$ ), but also the *post-Keplerian* parameters associated with the relativistic effects mentioned above. For example, effect (i) discussed above is parameterized by a quantity denoted  $\gamma_T$ ; effect (ii) by (among others) the quantities  $\dot{\omega}$ ,  $\dot{P}$ ; effect (iii) by the quantities  $r$ ,  $s$ ; etc.

The way in which observations of binary pulsars allow one to test relativistic theories of gravity is therefore the following. A (least-squares) fit between the observational timing data,  $\Delta_N^{\text{obs}}$ , and the parameterized theoretical timing formula,  $\Delta_N^{\text{th}}(P, x_A, e; \gamma_T, \dot{\omega}, \dot{P}, r, s)$ , allows for the determination of the observational values of the Keplerian ( $P^{\text{obs}}, x_A^{\text{obs}}, e^{\text{obs}}$ ) and post-Keplerian ( $\gamma_T^{\text{obs}}, \dot{\omega}^{\text{obs}}, \dot{P}^{\text{obs}}, r^{\text{obs}}, s^{\text{obs}}$ ) parameters. The theory of general relativity predicts the value of each post-Keplerian parameter as a function of the Keplerian parameters and the two masses of the binary system (the mass  $m_A$  of the pulsar and the mass  $m_B$  of the companion). For example, the theoretical value predicted by general relativity for the parameter  $\gamma_T$  is  $\gamma_T^{\text{GR}}(m_A, m_B) = en^{-1}(GMn/c^3)^{2/3} m_B(m_A + 2m_B)/M^2$ , where  $e$  is the eccentricity,  $n = 2\pi/P$  the orbital frequency, and  $M \equiv m_A + m_B$ . We thus see that, if one assumes that general relativity is correct, the observational measurement of a post-Keplerian parameter, for example  $\gamma_T^{\text{obs}}$ , determines a *curve* in the plane  $(m_A, m_B)$  of the two masses:  $\gamma_T^{\text{GR}}(m_A, m_B) = \gamma_T^{\text{obs}}$ , in our example. The measurement of two post-Keplerian parameters thus gives two curves in the  $(m_A, m_B)$  plane and generically allows one to determine the values of the two masses  $m_A$  and  $m_B$ , by considering the intersection of the two curves. We obtain a test of general relativity as soon as one observationally measures three or more post-Keplerian parameters: if the three (or more) curves all intersect at one point in the plane of the two masses, the theory of general relativity is confirmed, but if this is not the case the theory is refuted. At present, four distinct binary pulsars have allowed one to test general relativity. These four “relativistic” binary pulsars are: the first binary pulsar PSR B1913+16, the pulsar PSR B1534+12 (discovered by A. Wolszczan in 1991), and two recently discovered pulsars: PSR J1141–6545 (discovered in 1999 by V.M. Kaspi et al., whose first timing results

are due to M. Bailes et al. in 2003), and PSR J0737–3039 (discovered in 2003 by M. Burgay et al., whose first timing results are due to A.G. Lyne et al. and M. Kramer et al.). With the exception of PSR J1141–6545, whose companion is a white dwarf, the companions of the pulsars are neutron stars. In the case of PSR J0737–3039 the companion turns out to also be a pulsar that is visible from Earth.

In the system PSR B1913+16, *three* post-Keplerian parameters have been measured ( $\dot{\omega}, \gamma_T, \dot{P}$ ), which gives *one* test of the theory. In the system PSR J1141–65, *three* post-Keplerian parameters have been measured ( $\dot{\omega}, \gamma_T, \dot{P}$ ), which gives *one* test of the theory. (The parameter  $s$  is also measured through scintillation phenomena, but the use of this measurement for testing gravitation is more problematic.) In the system PSR B1534+12, *five* post-Keplerian parameters have been measured, which gives *three* tests of the theory. In the system PSR J0737–3039, *six* post-Keplerian parameters,<sup>4</sup> which gives *four* tests of the theory. It is remarkable that all of these tests have confirmed general relativity. See Figure 4 and, for references and details, [4, 11, 12, 13], as well as the contribution by Michael Kramer.

Note that, in Figure 4, some post-Keplerian parameters are measured with such great precision that they in fact define very thin curves in the  $m_A, m_B$  plane. On the other hand, some of them are only measured with a rough fractional precision and thus define “thick curves,” or “strips” in the plane of the masses (see, for example, the strips associated with  $\dot{P}$ ,  $r$  and  $s$  in the case of PSR B1534+12). In any case, the theory is confirmed when all of the strips (thick or thin) have a non-empty common intersection. (One should also note that the strips represented in Figure 4 only use the “one sigma” error bars, in other words a 68% level of confidence. Therefore, the fact that the  $\dot{P}$  strip for PSR B1534+12 is a little bit disjoint from the intersection of the other strips is not significant: a “two sigma” figure would show excellent agreement between observation and general relativity.)

In view of the arguments presented above, all of the tests shown in Figure 4 confirm the validity of general relativity in the regime of strong gravitational fields ( $h_{\mu\nu} \sim 1$ ). Moreover, the four tests that use measurements of the parameter  $\dot{P}$  (in the four corresponding systems) are direct experimental confirmations of the fact that the gravitational interaction propagates at the speed  $c$  between the companion and the pulsar. In fact,  $\dot{P}$  denotes the long-term variation  $\langle dP/dt \rangle$  of the orbital period. Detailed theoretical calculations of the motion of two gravitationally condensed objects in general relativity, that take into account the effects connected to the propagation of the gravitational interaction at finite speed[14], have shown that one of the observable effects of this propagation is a long-term decrease in the orbital period given by the formula

$$\dot{P}^{\text{GR}}(m_A, m_B) = -\frac{192\pi}{5} \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1 - e^2)^{7/2}} \left( \frac{GMn}{c^3} \right)^{5/3} \frac{m_A m_B}{M^2}.$$

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<sup>4</sup>In the case of PSR J0737–3039, one of the six measured parameters is the ratio  $x_A/x_B$  between a Keplerian parameter of the pulsar and its analog for the companion, which turns out to also be a pulsar.



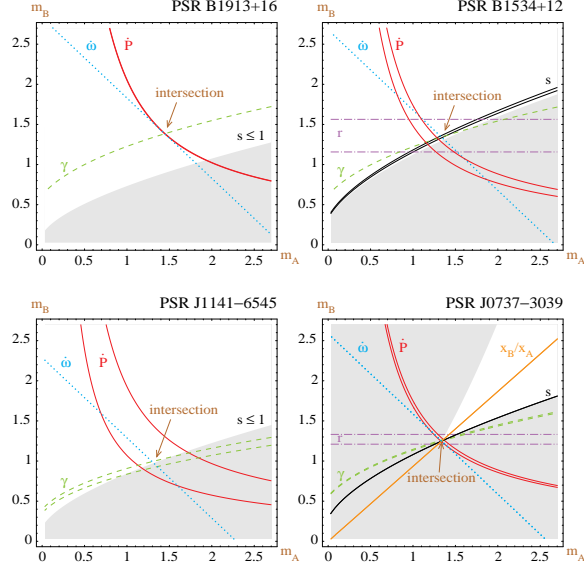


Figure 4: Tests of general relativity obtained from observations of four binary pulsars. For each binary pulsar one has traced the “curves,” in the plane of the two masses ( $m_A$  = mass of the pulsar,  $m_B$  = mass of the companion), defined by equating the theoretical expressions for the various post-Keplerian parameters, as predicted by general relativity, to their observational value, determined through a least-squares fit to the parameterized theoretical timing formula. Each “curve” is in fact a “strip,” whose thickness is given by the (one sigma) precision with which the corresponding post-Keplerian parameter is measured. For some parameters, these strips are too thin to be visible. The grey zones would correspond to a sine for the angle of inclination of the orbital plane with respect to the plane of the sky that is greater than 1, and are therefore physically excluded.

The direct physical origin of this decrease in the orbital period lies in the modification, produced by general relativity, of the usual Newtonian law of gravitational attraction between two bodies,  $F_{\text{Newton}} = G m_A m_B / r_{AB}^2$ . In place of such a simple law, general relativity predicts a more complicated force law that can be expanded in the symbolic form

$$F_{\text{Einstein}} = \frac{G m_A m_B}{r_{AB}^2} \left( 1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \frac{v^5}{c^5} + \frac{v^6}{c^6} + \frac{v^7}{c^7} + \dots \right), \quad (20)$$

where, for example, “ $v^2/c^2$ ” represents a whole set of terms of order  $v_A^2/c^2$ ,  $v_B^2/c^2$ ,  $v_A v_B/c^2$ , or even  $G m_A/c^2 r$  or  $G m_B/c^2 r$ . Here  $v_A$  denotes the speed of body  $A$ ,  $v_B$  that of body  $B$ , and  $r_{AB}$  the distance between the two bodies. The term of order  $v^5/c^5$  in Equation (20) is particularly important. This

term is a direct consequence of the finite-speed propagation of the gravitational interaction between  $A$  and  $B$ , and its calculation shows that it contains a component that is opposed to the relative speed  $\mathbf{v}_A - \mathbf{v}_B$  of the two bodies and that, consequently, slows down the orbital motion of each body, causing it to evolve towards an orbit that lies closer to its companion (and therefore has a shorter orbital period). This “braking” term (which is correlated with the emission of gravitational waves),  $\delta F_{\text{Einstein}} \sim v^5/c^5 F_{\text{Newton}}$ , leads to a long-term decrease in the orbital period  $\dot{P}^{\text{GR}} \sim -(v/c)^5 \sim -10^{-12}$  that is very small, but whose reality has been verified with a fractional precision of order  $10^{-3}$  in PSR B1913+16 and of order 20% in PSR B1534+12 and PSR J1141–6545 [4, 11, 13].

To conclude this brief outline of the tests of relativistic gravitation by binary pulsars, let us note that there is an analog, for the regime of strong gravitational fields, of the formalism of parametrization for possible deviations from general relativity mentioned in Section 6 in the framework of weak gravitational fields (using the post-Newtonian parameters  $\gamma - 1$  and  $\beta - 1$ ). This analog is obtained by considering a two-parameter family of relativistic theories of gravitation, assuming that the gravitational interaction is propagated not only by a tensor field  $g_{\mu\nu}$  but also by a scalar field  $\varphi$ . Such a class of tensor-scalar theories of gravitation allows for a description of possible deviations in both the solar system and in binary pulsars. It also allows one to explicitly demonstrate that binary pulsars indeed test the effects of strong fields that go beyond the tests of the weak fields of the solar system by exhibiting classes of theories that are compatible with all of the observations in the solar system but that are incompatible with the observations of binary pulsars, see [4, 13].

## 10 Gravitational Waves: Propagation, Generation, and Detection

As soon as he had finished constructing the theory of general relativity, Einstein realized that it implied the existence of waves of geometric deformations of space-time, or “gravitational waves” [15, 2]. Mathematically, these waves are analogs (with the replacement  $A_\mu \rightarrow h_{\mu\nu}$ ) of electromagnetic waves, but conceptually they signify something remarkable: they exemplify, in the purest possible way, the “elastic” nature of space-time in general relativity. Before Einstein space-time was a rigid structure, given a priori, which was not influenced by the material content of the Universe. After Einstein, a distribution of matter (or more generally of mass-energy) that changes over the course of time, let us say for concreteness a binary system of two neutron stars or two black holes, will not only deform the chrono-geometry of the space-time in its immediate neighborhood, but this deformation will propagate in every possible direction away from the system considered, and will travel out to infinity in the form of a wave whose oscillations will reflect the temporal variations of the matter distribution. We therefore see that the study of these gravitational waves poses three separate problems: that of generation, that of propagation, and, finally, that of

detection of such gravitational radiation. These three problems are at present being actively studied, since it is hoped that we will soon detect gravitational waves, and thus will be able to obtain new information about the Universe [16]. We shall here content ourselves with an elementary introduction to this field of research. For a more detailed introduction to the detection of gravitational waves see the contribution by Jean-Yves Vinet to this Poincaré seminar.

Let us first consider the simplest case of very weak gravitational waves, outside of their material sources. The geometry of such a space-time can be written, as in Section 6, as  $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$ , where  $h_{\mu\nu} \ll 1$ . At first order in  $h$ , and outside of the source (namely in the domain where  $T_{\mu\nu}(x) = 0$ ), the perturbation of the geometry,  $h_{\mu\nu}(x)$ , satisfies a homogeneous equation obtained by replacing the right-hand side of Equation (11) with zero. It can be shown that one can simplify this equation through a suitable choice of coordinate system. In a *transverse traceless* (TT) coordinate system the only non-zero components of a general gravitational wave are the spatial components  $h_{ij}^{\text{TT}}$ ,  $i, j = 1, 2, 3$  (in other words  $h_{00}^{\text{TT}} = 0 = h_{0i}^{\text{TT}}$ ), and these components satisfy

$$\square h_{ij}^{\text{TT}} = 0, \quad \partial_j h_{ij}^{\text{TT}} = 0, \quad h_{jj}^{\text{TT}} = 0. \quad (21)$$

The first equation in (21), where the wave operator  $\square = \Delta - c^{-2} \partial_t^2$  appears, shows that gravitational waves (like electromagnetic waves) propagate at the speed  $c$ . If we consider for simplicity a monochromatic plane wave ( $h_{ij}^{\text{TT}} = \zeta_{ij} \exp(i \mathbf{k} \cdot \mathbf{x} - i \omega t) + \text{complex conjugate}$ , with  $\omega = c |\mathbf{k}|$ ), the second equation in (21) shows that the (complex) tensor  $\zeta_{ij}$  measuring the polarization of a gravitational wave only has non-zero components in the plane orthogonal to the wave's direction of propagation:  $\zeta_{ij} k^j = 0$ . Finally, the third equation in (21) shows that the polarization tensor  $\zeta_{ij}$  has vanishing trace:  $\zeta_{jj} = 0$ . More concretely, this means that if a gravitational wave propagates in the  $z$ -direction, its polarization is described by a  $2 \times 2$  matrix,  $\begin{pmatrix} \zeta_{xx} & \zeta_{xy} \\ \zeta_{yx} & \zeta_{yy} \end{pmatrix}$ , which is symmetric and traceless. Such a polarization matrix therefore only contains two independent (complex) components:  $\zeta_+ \equiv \zeta_{xx} = -\zeta_{yy}$ , and  $\zeta_\times \equiv \zeta_{xy} = \zeta_{yx}$ . This is the same number of independent (complex) components that an electromagnetic wave has. Indeed, in a transverse gauge, an electromagnetic wave only has spatial components  $A_i^T$  that satisfy

$$\square A_i^T = 0, \quad \partial_j A_j^T = 0. \quad (22)$$

As in the case above, the first equation (22) means that an electromagnetic wave propagates at the speed  $c$ , and the second equation shows that a monochromatic plane electromagnetic wave ( $A_i^T = \zeta_i \exp(i \mathbf{k} \cdot \mathbf{x} - i \omega t) + \text{c.c.}$ ,  $\omega = c |\mathbf{k}|$ ) is described by a (complex) polarization vector  $\zeta_i$  that is orthogonal to the direction of propagation:  $\zeta_j k^j = 0$ . For a wave propagating in the  $z$ -direction such a vector only has two independent (complex) components,  $\zeta_x$  and  $\zeta_y$ . It is indeed the same number of components that a gravitational wave has, but we see that the two quantities measuring the polarization of a gravitational wave,  $\zeta_+ = \zeta_{xx} = -\zeta_{yy}$ ,  $\zeta_\times = \zeta_{xy} = \zeta_{yx}$  are mathematically quite different from

the two quantities  $\zeta_x, \zeta_y$  measuring the polarization of an electromagnetic wave. However, see Section 11 below.

We have here discussed the propagation of a gravitational wave in a background space-time described by the Minkowski metric  $\eta_{\mu\nu}$ . One can also consider the propagation of a wave in a curved background space-time, namely by studying solutions of Einstein's equations (9) of the form  $g_{\mu\nu}(x) = g_{\mu\nu}^B(x) + h_{\mu\nu}(x)$  where  $h_{\mu\nu}$  is not only small, but varies on temporal and spatial scales much shorter than those of the background metric  $g_{\mu\nu}^B(x)$ . Such a study is necessary, for example, for understanding the propagation of gravitational waves in the cosmological Universe.

The problem of *generation* consists in searching for the connection between the tensorial amplitude  $h_{ij}^{\text{TT}}$  of the gravitational radiation in the radiation zone and the motion and structure of the source. If one considers the simplest case of a source that is sufficiently diffuse that it only creates waves that are everywhere weak ( $g_{\mu\nu} - \eta_{\mu\nu} = h_{\mu\nu} \ll 1$ ), one can use the linearized approximation to Einstein's equations (9), namely Equations (11). One can solve Equations (11) by the same technique that is used to solve Maxwell's equations (12): one fixes the coordinate system by imposing  $\partial^\alpha h_{\alpha\mu} - \frac{1}{2} \partial_\mu h^\alpha_\alpha = 0$  (analogous to the Lorentz gauge condition  $\partial^\alpha A_\alpha = 0$ ), then one inverts the wave operator by using retarded potentials. Finally, one must study the asymptotic form, at infinity, of the emitted wave, and write it in the reduced form of a transverse and traceless amplitude  $h_{ij}^{\text{TT}}$  satisfying Equations (21) (analogous to a transverse electromagnetic wave  $A_i^T$  satisfying (22)). One then finds that, just as charge conservation implies that there is no monopole type electro-magnetic radiation, but only dipole or higher orders of polarity, the conservation of energy-momentum implies the absence of monopole *and* dipole gravitational radiation. For a slowly varying source ( $v/c \ll 1$ ), the dominant gravitational radiation is of *quadrupole* type. It is given, in the radiation zone, by an expression of the form

$$h_{ij}^{\text{TT}}(t, r, \mathbf{n}) \simeq \frac{2G}{c^4 r} \frac{\partial^2}{\partial t^2} [I_{ij}(t - r/c)]^{\text{TT}}. \quad (23)$$

Here  $r$  denotes the distance to the center of mass of the source,  $I_{ij}(t) \equiv \int d^3x c^{-2} T^{00}(t, \mathbf{x}) (x^i x^j - \frac{1}{3} \mathbf{x}^2 \delta^{ij})$  is the quadrupole moment of the mass-energy distribution, and the upper index TT denotes an algebraic projection operation for the quadrupole tensor  $I_{ij}$  (which is a  $3 \times 3$  matrix) that only retains the part orthogonal to the local direction of wave propagation  $n^i \equiv x^i/r$  with vanishing trace ( $I_{ij}^{\text{TT}}$  is therefore locally a (real)  $2 \times 2$  symmetric, traceless matrix of the same type as  $\zeta_{ij}$  above). Formula (23) (which was in essence obtained by Einstein in 1918 [15]) is only the first approximation to an expansion in powers of  $v/c$ , where  $v$  designates an internal speed characteristic of the source. The prospect of soon being able to detect gravitational waves has motivated theorists to improve Formula (23): (i) by describing the terms of higher order in  $v/c$ , up to a very high order, and (ii) by using new approximation methods that allow one to treat sources containing regions of strong gravitational fields (such as, for example, a binary system of two black holes or two neutron stars). See below for the most recent results.

Finally, the problem of *detection*, of which the pioneer was Joseph Weber in the 1960s, is at present giving rise to very active experimental research. The principle behind any detector is that a gravitational wave of amplitude  $h_{ij}^{\text{TT}}$  induces a change in the distance  $L$  between two bodies on the order of  $\delta L \sim hL$  during its passage. One way of seeing this is to consider the action of a wave  $h_{ij}^{\text{TT}}$  on two free particles, at rest before the arrival of the wave at the positions  $x_1^i$  and  $x_2^i$  respectively. As we have seen, each particle, in the presence of the wave, will follow a geodesic motion in the geometry  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$  (with  $h_{00} = h_{0i} = 0$  and  $h_{ij} = h_{ij}^{\text{TT}}$ ). By writing out the geodesic equation, Equation (7), one finds that it simply reduces (at first order in  $h$ ) to  $d^2x^i/ds^2 = 0$ . Therefore, particles that are initially at rest ( $\dot{x}^i = \text{const.}$ ) remain at rest in a transverse and traceless system of coordinates! This does not however mean that the gravitational wave has no observable effect. In fact, since the spatial geometry is perturbed by the passage of the wave,  $g_{ij}(t, \mathbf{x}) = \delta_{ij} + h_{ij}^{\text{TT}}(t, \mathbf{x})$ , one finds that the physical distance between the two particles  $x_1^i, x_2^i$  (which is observable, for example, by measuring the time taken for light to make a round trip between the two particles) varies, during the passage of the wave, according to  $L^2 = (\delta_{ij} + h_{ij}^{\text{TT}})(x_2^i - x_1^i)(x_2^j - x_1^j)$ .

The problem of detecting a gravitational wave thus leads to the problem of detecting a small relative displacement  $\delta L/L \sim h$ . By using Formula (23), one finds that the order of magnitude of  $h$ , for known or hoped for astrophysical sources (for example, a very close system of two neutron stars or two black holes), situated at distances such that one may hope to see several events per year ( $r \gtrsim 600$  million light-years), is in fact extremely small:  $h \lesssim 10^{-22}$  for signals whose characteristic frequency is around 100 Hertz. Several types of detectors have been developed since the pioneering work of J. Weber [16]. At present, the detectors that should succeed in the near future at detecting amplitudes  $h \sim \delta L/L \sim 10^{-22}$  are large interferometers, of the Michelson or Fabry-Pérot type, having arms that are many kilometers in length into which a very powerful monochromatic laser beam is injected. Such terrestrial interferometric detectors presently exist in the U.S.A. (the LIGO detectors [17]), in Europe (the VIRGO [18] and GEO 600 [19] detectors) and elsewhere (such as the TAMA detector in Japan). Moreover, the international space project LISA [20], made up of an interferometer between satellites that are several million kilometers apart, should allow one to detect low frequency ( $\sim$  one hundredth or one thousandth of a Hertz) gravitational waves in a dozen years or so. This collection of gravitational wave detectors promises to bring invaluable information for astronomy by opening a new “window” on the Universe that is much more transparent than the various electromagnetic (or neutrino) windows that have so greatly expanded our knowledge of the Universe in the twentieth century.

The extreme smallness of the expected gravitational signals has led a number of experimentalists to contribute, over many years, a wealth of ingenuity and know-how in order to develop technology that is sufficiently precise and trustworthy (see [17, 18, 19, 20]). To conclude, let us also mention how much concerted theoretical effort has been made, both in calculating the general rel-

ativistic predictions for gravitational waves emitted by certain sources, and in developing methods adapted to the extraction of the gravitational signal from the background noise in the detectors. For example, one of the most promising sources for terrestrial detectors is the wave train for gravitational waves emitted by a system of two black holes, and in particular the final (most intense) portion of this wave train, which is emitted during the last few orbits of the system and the final coalescence of the two black holes into a single, more massive black hole. We have seen above (see Section 9) that the finite speed of propagation of the gravitational interaction between the two bodies of a binary system gives rise to a progressive acceleration of the orbital frequency, connected to the progressive approach of the two bodies towards each other. Here we are speaking of the final stages in such a process, where the two bodies are so close that they orbit around each other in a spiral pattern that accelerates until they attain (for the final “stable” orbits) speeds that become comparable to the speed of light, all the while remaining slightly slower. In order to be able to determine, with a precision that is acceptable for the needs of detection, the dynamics of such a binary black hole system in such a situation, as well as the gravitational amplitude  $h_{ij}^{\text{TT}}$  that it emits, it was necessary to develop a whole ensemble of analytic techniques to a very high level of precision. For example, it was necessary to calculate the expansion (20) of the force determining the motion of the two bodies to a very high order and also to calculate the amplitude  $h_{ij}^{\text{TT}}$  of the gravitational radiation emitted to infinity with a precision going well beyond the quadrupole approximation (23). These calculations are comparable in complexity to high-order calculations in quantum field theory. Some of the techniques developed for quantum field theory indeed proved to be extremely useful for these calculations in the (classical) theory of general relativity (such as certain resummation methods and the mathematical use of analytic continuation in the number of space-time dimensions). For an entryway into the literature of these modern analytic methods, see [21], and for an early example of a result obtained by such methods of direct interest for the physics of detection see Figure 5 [22], which shows a component of the gravitational amplitude  $h_{ij}^{\text{TT}}(t)$  emitted during the final stages of evolution of a system of two black holes of equal mass. The first oscillations shown in Figure 5 are emitted during the last quasi-circular orbits (accelerated motion in a spiral of decreasing radius). The middle part of the signal corresponds to a phase where, having moved past the last stable orbit, the two black holes “fall” toward each other while spiraling rapidly. In fact, contrarily to Newton’s theory, which predicts that two condensed bodies would be able to orbit around each other with an orbit of arbitrarily small radius (basically up until the point that the two bodies touch), Einstein’s theory predicts a modified law for the force between the two bodies, Equation (20), whose analysis shows that it is so attractive that it no longer allows for stable circular orbits when the distance between the two bodies becomes smaller than around  $6 G(m_A + m_B)/c^2$ . In the case of two black holes, this distance is sufficiently larger than the black hole “radii” ( $2 G m_A/c^2$  and  $2 G m_B/c^2$ ) that one is still able to analytically treat the beginning of the “spiralling plunge” of the two black holes towards each other. The final oscillations in Figure 5 are

emitted by the rotating (and initially highly deformed) black hole formed from the merger of the two initial, separate black holes.

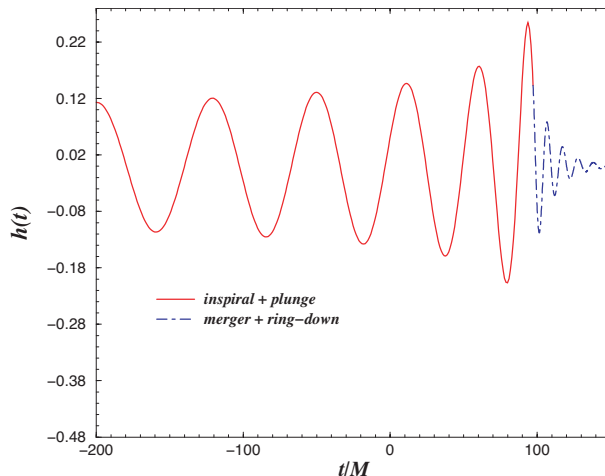


Figure 5: The gravitational amplitude  $h(t)$  emitted during the final stages of evolution of a system of two equal-mass black holes. The beginning of the signal (the left side of the figure), which is sinusoidal, corresponds to an inspiral motion of two separate black holes (with decreasing distance); the middle corresponds to a rapid “inspiralling plunge” of the two black holes towards each other; the end (at right) corresponds to the oscillations of the final, rotating black hole formed from the merger of the two initial black holes.

Up until quite recently the analytic predictions illustrated in Figure 5 concerning the gravitational signal  $h(t)$  emitted by the spiralling plunge and merger of two black holes remained conjectural, since they could be compared to neither other theoretical predictions nor to observational data. Recently, worldwide efforts made over three decades to attack the problem of the coalescence of two black holes by *numerically* solving Einstein’s equations (9) have spectacularly begun to bear fruit. Several groups have been able to numerically calculate the signal  $h(t)$  emitted during the final orbits and merger of two black holes [23]. In essence, there is good agreement between the analytical and numerical predictions. In order to be able to detect the gravitational waves emitted by the coalescence of two black holes, it will most likely be necessary to properly combine the information on the structure of the signal  $h(t)$  obtained by the two types of methods, which are in fact complementary.

## 11 General Relativity and Quantum Theory: From Supergravity to String Theory

Up until now, we have discussed the *classical* theory of general relativity, neglecting any quantum effects. What becomes of the theory in the quantum regime? This apparently innocent question in fact opens up vast new prospects that are still under construction. We will do nothing more here than to touch upon the subject, by pointing out to the reader some of the paths along which contemporary physics has been led by the challenge of unifying general relativity and quantum theory. For a more complete introduction to the various possibilities “beyond” general relativity suggested within the framework of *string theory* (which is still under construction) one should consult the contribution of Ignatios Antoniadis to this Poincaré Seminar.

Let us recall that, from the very beginning of the quasi-definitive formulation of quantum theory (1925–1930), the creators of quantum mechanics (Born, Heisenberg, Jordan; Dirac; Pauli; etc.) showed how to “quantize” not only systems with several particles (such as an atom), but also *fields*, continuous dynamical systems whose classical description implies a continuous distribution of energy and momentum in space. In particular, they showed how to *quantize* (or in other words how to formulate within a framework compatible with quantum theory) the electromagnetic field  $A_\mu$ , which, as we have recalled above, satisfies the Maxwell equations (12) at the classical level. They nevertheless ran into difficulty due to the following fact. In quantum theory, the physics of a system’s evolution is essentially contained in the *transition amplitudes*  $A(f, i)$  between an initial state labelled by  $i$  and a final state labelled by  $f$ . These amplitudes  $A(f, i)$  are complex numbers. They satisfy a “transitivity” property of the type

$$A(f, i) = \sum_n A(f, n) A(n, i), \quad (24)$$

which contains a sum over all possible intermediate states, labelled by  $n$  (with this sum becoming an integral when there is a continuum of intermediate possible states). R. Feynman used Equation (24) as a point of departure for a new formulation of quantum theory, by interpreting it as an analog of *Huygens’ Principle*: if one thinks of  $A(f, i)$  as the amplitude, “at the point  $f$ ,” of a “wave” emitted “from the point  $i$ ,” Equation (24) states that this amplitude can be calculated by considering the “wave” emitted from  $i$  as passing through all possible intermediate “points”  $n$  ( $A(n, i)$ ), while reemitting “wavelets” starting from these intermediate points ( $A(f, n)$ ), which then superpose to form the total wave arriving at the “final point  $f$ .”

Property (24) does not pose any problem in the quantum mechanics of discrete systems (particle systems). It simply shows that the amplitude  $A(f, i)$  behaves like a wave, and therefore must satisfy a “wave equation” (which is indeed the case for the Schrödinger equation describing the dependence of  $A(f, i)$  on the parameters determining the final configuration  $f$ ). On the other hand, Property (24) poses formidable problems when one applies it to the quantiza-



tion of continuous dynamical systems (fields). In fact, for such systems the “space” of intermediate possible states is infinitely larger than in the case of the mechanics of discrete systems. Roughly speaking, the intermediate possible states for a field can be described as containing  $\ell = 1, 2, 3, \dots$  quantum excitations of the field, with each quantum excitation (or pair of “virtual particles”) being described essentially by a plane wave,  $\zeta \exp(i k_\mu x^\mu)$ , where  $\zeta$  measures the polarization of these virtual particles and  $k^\mu = \eta^{\mu\nu} k_\nu$ , with  $k^0 = \omega$  and  $k^i = \mathbf{k}$ , their angular frequency and wave vector, or (using the Planck-Einstein-de Broglie relations  $E = \hbar\omega$ ,  $\mathbf{p} = \hbar\mathbf{k}$ ) their energy-momentum  $p^\mu = \hbar k^\mu$ . The quantum theory shows (basically because of the uncertainty principle) that the four-frequencies (and four-momenta)  $p^\mu = \hbar k^\mu$  of the intermediate states cannot be constrained to satisfy the classical equation  $\eta_{\mu\nu} p^\mu p^\nu = -m^2$  (or in other words  $E^2 = \mathbf{p}^2 + m^2$ ; we use  $c = 1$  in this section). As a consequence, the sum over intermediate states for a quantum field theory has the following properties (among others): (i) when  $\ell = 1$  (an intermediate state containing only one pair of virtual particles, called a *one-loop contribution*), there is an integral over a four-momentum  $p^\mu$ ,  $\int d^4p = \int dE \int d\mathbf{p}$ ; (ii) when  $\ell = 2$  (two pairs of virtual particles; a *two-loop contribution*), there is an integral over two four-momenta  $p_1^\mu, p_2^\mu$ ,  $\int d^4p_1 d^4p_2$ ; etc. The delicate point comes from the fact that the energy-momentum of an intermediate state can take arbitrarily high values. This possibility is directly connected (through a Fourier transform) to the fact that a field possesses an infinite number of degrees of freedom, corresponding to configurations that vary over arbitrarily small time and length scales.

The problems posed by the necessity of integrating over the infinite domain of four-momenta of intermediate virtual particles (or in other words of accounting for the fact that field configurations can vary over arbitrarily small scales) appeared in the 1930s when the quantum theory of the electromagnetic field  $A_\mu$  (called quantum electrodynamics, or QED) was studied in detail. These problems imposed themselves in the following form: when one calculates the transition amplitude for given initial and final states (for example the collision of two light quanta, with two photons entering and two photons leaving) by using (24), one finds a result given in the form of a *divergent integral*, because of the integral (in the one-loop approximation,  $\ell = 1$ ) over the arbitrarily large energy-momentum describing virtual electron-positron pairs appearing as possible intermediate states. Little by little, theoretical physicists understood that the types of divergent integrals appearing in QED were relatively benign and, after the second world war, they developed a method (*renormalization theory*) that allowed one to unambiguously isolate the infinite part of these integrals, and to subtract them by expressing the amplitudes  $A(f, i)$  solely as a function of observable quantities [24] (work by J. Schwinger, R. Feynman, F. Dyson etc.).

The preceding work led to the development of consistent quantum theories not only for the electromagnetic field  $A_\mu$  (QED), but also for generalizations of electromagnetism (Yang-Mills theory or non-abelian gauge theory) that turned out to provide excellent descriptions of the new interactions between elementary particles discovered in the twentieth century (the electroweak theory, partially unifying electromagnetism and weak nuclear interactions, and quantum chro-

modynamics, describing the strong nuclear interactions). All of these theories give rise to only relatively benign divergences that can be “renormalized” and thus allowed one to compute amplitudes  $A(f, i)$  corresponding to observable physical processes [24] (notably, work by G. ’t Hooft and M. Veltman).

What happens when we use (24) to construct a “perturbative” quantum theory of general relativity (namely one obtained by expanding in the number  $\ell$  of virtual particle pairs appearing in the intermediate states)? The answer is that the integrals over the four-momenta of intermediate virtual particles are not at all of the benign type that allowed them to be renormalized in the simpler case of electromagnetism. The source of this difference is not accidental, but is rather connected with the basic physics of relativistic gravitation. Indeed, as we have mentioned, the virtual particles have arbitrarily large energies  $E$ . Because of the basic relations that led Einstein to develop general relativity, namely  $E = m_i$  and  $m_i = m_g$ , one deduces that these virtual particles correspond to arbitrarily large gravitational masses  $m_g$ . They will therefore end up creating intense gravitational effects that become more and more intense as the number  $\ell$  of virtual particle pairs grows. These gravitational interactions that grow without limit with energy and momentum correspond (by Fourier transform) to field configurations concentrated in arbitrarily small space and time scales. One way of seeing why the quantum gravitational field creates much more violent problems than the quantum electromagnetic field is, quite simply, to go back to dimensional analysis. Simple considerations in fact show that the relative (non-dimensional) one-loop amplitude  $A_1$  must be proportional to the product  $\hbar G$  and must contain an integral  $\int d^4k$ . However, in 1900 Planck had noticed that (in units where  $c = 1$ ) the dimensions of  $\hbar$  and  $G$  were such that the product  $\hbar G$  had the dimensions of length (or time) squared:

$$\ell_P \equiv \sqrt{\frac{\hbar G}{c^3}} \simeq 1.6 \times 10^{-33} \text{ cm}, \quad t_P \equiv \sqrt{\frac{\hbar G}{c^5}} \simeq 5.4 \times 10^{-44} \text{ s}. \quad (25)$$

One thus deduces that the integral  $\int d^4k f(k)$  must have the dimensions of a squared frequency, and therefore that  $A_1$  must (when  $k \rightarrow \infty$ ) be of the type,  $A_1 \sim \hbar G \int d^4k / k^2$ . Such an integral diverges quadratically with the upper limit  $\Lambda$  of the integral (the cutoff frequency, such that  $|k| \leq \Lambda$ ), so that  $A_1 \sim \hbar G \Lambda^2 \sim t_P^2 \Lambda^2$ . The extension of this dimensional analysis to the intermediate states with several loops ( $\ell > 1$ ) causes even more severe polynomial divergences to appear, of a type such that the power of  $\Lambda$  that appears grows without limit with  $\ell$ .

In summary, the essential physical characteristics of gravitation ( $E = m_i = m_g$  and the dimension of Newton’s constant  $G$ ) imply the impossibility of generalizing to the gravitational case the methods that allowed a satisfactory quantum treatment of the other interactions (electromagnetic, weak, and strong). Several paths have been explored to get out of this impasse. Some researchers tried to quantize general relativity non-perturbatively, without using an expansion in intermediate states (24) (work by A. Ashtekar, L. Smolin, and others). Others have tried to generalize general relativity by adding a fermionic field to Einstein’s

(bosonic) gravitational field  $g_{\mu\nu}(x)$ , the gravitino field  $\psi_\mu(x)$ . It is indeed remarkable that it is possible to define a theory, known as *supergravity*, that generalizes the geometric invariance of general relativity in a profound way. After the 1974 discovery (by J. Wess and B. Zumino) of a possible new global symmetry for interacting bosonic and fermionic fields, *supersymmetry* (which is a sort of global rotation transforming bosons to fermions and vice versa), D.Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara; and S. Deser and B. Zumino; showed that one could generalize global supersymmetry to a *local supersymmetry*, meaning that it varies from point to point in space-time. Local supersymmetry is a sort of fermionic generalization (with anti-commuting parameters) of the geometric invariance at the base of general relativity (the invariance under any change in coordinates). The generalization of Einstein's theory of gravitation that admits such a local supersymmetry is called *supergravity theory*. As we have mentioned, in four dimensions this theory contains, in addition to the (commuting) bosonic field  $g_{\mu\nu}(x)$ , an (anti-commuting) fermionic field  $\psi_\mu(x)$  that is both a space-time vector (with index  $\mu$ ) and a spinor. (It is a massless field of spin  $3/2$ , intermediate between a massless spin 1 field like  $A_\mu$  and a massless spin 2 field like  $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ .) Supergravity was extended to richer fermionic structures (with many gravitinos), and was formulated in space-times having more than four dimensions. It is nevertheless remarkable that there is a maximal dimension, equal to  $D = 11$ , admitting a theory of supergravity (the maximal supergravity constructed by E. Cremmer, B. Julia, and J. Scherk). The initial hope underlying the construction of these supergravity theories was that they would perhaps allow one to give meaning to the perturbative calculation (24) of quantum amplitudes. Indeed, one finds for example that at one loop,  $\ell = 1$ , the contributions coming from intermediate fermionic states have a sign opposite to the bosonic contributions and (because of the supersymmetry, bosons  $\leftrightarrow$  fermions) exactly cancel them. Unfortunately, although such cancellations exist for the lowest orders of approximation, it appeared that this was probably not going to be the case at all orders<sup>5</sup>. The fact that the gravitational interaction constant  $G$  has "a bad dimension" remains true and creates non-renormalizable divergences starting at a certain number of loops  $\ell$ .

Meanwhile, a third way of defining a consistent quantum theory of gravity was developed, under the name of *string theory*. Initially formulated as models for the strong interactions (in particular by G. Veneziano, M. Virasoro, P. Ramond, A. Neveu, and J.H. Schwarz), the string theories were founded upon the quantization of the relativistic dynamics of an extended object of one spatial dimension: a "string." This string could be closed in on itself, like a small rubber band (a closed string), or it could have two ends (an open string). Note that the point of departure of string theory only includes the Poincaré-Minkowski space-time, in other words the metric  $\eta_{\mu\nu}$  of Equation (2), and quantum theory (with the constant  $\hbar = h/2\pi$ ). In particular, the only symmetry manifest in the classical dynamics of a string is the Poincaré group (3). It is, however, remark-

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<sup>5</sup>Recent work by Z. Bern et al. and M. Green et al., has, however, suggested that such cancellations take place at all orders for the case of maximal supergravity, dimensionally reduced to  $D = 4$  dimensions.

able that (as shown by T. Yoneya, and J. Scherk and J.H. Schwarz, in 1974) one of the quantum excitations of a closed string reproduces, in a certain limit, all of the non-linear structure of general relativity (see below). Among the other remarkable properties of string theory [25], let us point out that it is the first physical theory to determine the space-time dimension  $D$ . In fact, this theory is only consistent if  $D = 10$ , for the versions allowing fermionic excitations (the purely bosonic string theory selects  $D = 26$ ). The fact that  $10 > 4$  does not mean that this theory has no relevance to the real world. Indeed, it has been known since the 1930s (from work of T. Kaluza and O. Klein) that a space-time of dimension  $D > 4$  is compatible with experiment if the supplementary (spatial) dimensions close in on themselves (meaning they are *compactified*) on very small distance scales. The low-energy physics of such a theory seems to take place in a four-dimensional space-time, but it contains new (a priori massless) fields connected to the geometry of the additional compactified dimensions. Moreover, recent work (due in particular to I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali) has suggested the possibility that the additional dimensions are compactified on scales that are small with respect to everyday life, but very large with respect to the Planck length. This possibility opens up an entire phenomenological field dealing with the eventual observation of signals coming from string theory (see the contribution of I. Antoniadis to this Poincaré seminar).

However, string theory's most remarkable property is that it seems to avoid, in a radical way, the problems of divergent (non-renormalizable) integrals that have weighed down every direct attempt at perturbatively quantizing gravity. In order to explain how string theory arrives at such a result, we must discuss some elements of its formalism.

Recall that the classical dynamics of any system is obtained by minimizing a functional of the time evolution of the system's configuration, called the *action* (the principle of least action). For example, the action for a particle of mass  $m$ , moving in a Riemannian space-time (6), is proportional to the length of the line that it traces in space-time:  $S = -m \int ds$ . This action is minimized when the particle follows a geodesic, in other words when its equation of motion is given by (7). According to Y. Nambu and T. Goto, the action for a string is  $S = -T \iint dA$ , where the parameter  $T$  (analogous to  $m$  for the particle) is called the string *tension*, and where  $\iint dA$  is the area of the *two-dimensional* surface traced out by the evolution of the string in the ( $D$ -dimensional) space-time in which it lives. In quantum theory, the action functional serves (as shown by R. Feynman) to define the transition amplitude (24). Basically, when one considers two intermediate configurations  $m$  and  $n$  (in the sense of the right-hand side of (24)) that are close to each other, the amplitude  $A(n, m)$  is proportional to  $\exp(i S(n, m)/\hbar)$ , where  $S(n, m)$  is the minimal classical action such that the system considered evolves from the configuration labelled by  $n$  to that labelled by  $m$ . Generalizing the decomposition in (24) by introducing an infinite number of intermediate configurations that lie close to each other, one ends up (in a generalization of Huygens' principle) expressing the amplitude  $A(f, i)$  as a multiple sum over all of the "paths" (in the configuration space of

the system studied) connecting the initial state  $i$  to the final state  $f$ . Each path contributes a term  $e^{i\phi}$  where the phase  $\phi = S/\hbar$  is proportional to the action  $S$  corresponding to this “path,” or in other words to this possible evolution of the system. In string theory,  $\phi = -(T/\hbar) \iint dA$ . Since the phase is a non-dimensional quantity, and  $\iint dA$  has the dimension of an area, we see that the quantum theory of strings brings in the quantity  $\hbar/T$ , having the dimensions of a length squared, at a fundamental level. More precisely, the fundamental length of string theory,  $\ell_s$ , is defined by

$$\ell_s^2 \equiv \alpha' \equiv \frac{\hbar}{2\pi T}. \quad (26)$$

This fundamental length plays a central role in string theory. Roughly speaking, it defines the characteristic “size” of the quantum states of a string. If  $\ell_s$  is much smaller than the observational resolution with which one studies the string, the string will look like a point-like particle, and its interactions will be described by a quantum theory of relativistic particles, which is equivalent to a theory of relativistic fields. It is precisely in this sense that general relativity emerges as a limit of string theory. Since this is an important conceptual point for our story, let us give some details about the emergence of general relativity from string theory.

The action functional that is used in practice to quantize a string is not really  $-T \iint dA$ , but rather (as emphasized by A. Polyakov)

$$\frac{S}{\hbar} = -\frac{1}{4\pi\ell_s^2} \iint d^2\sigma \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \dots, \quad (27)$$

where  $\sigma^a$ ,  $a = 0, 1$  are two coordinates that allow an event to be located on the space-time surface (or ‘world-sheet’) traced out by the string within the ambient space-time;  $\gamma_{ab}$  is an auxiliary metric ( $d\Sigma^2 = \gamma_{ab}(\sigma) d\sigma^a d\sigma^b$ ) defined on this surface (with  $\gamma^{ab}$  being its inverse, and  $\gamma$  its determinant); and  $X^\mu(\sigma^a)$  defines the embedding of the string in the ambient (flat) space-time. The dots indicate additional terms, and in particular terms of fermionic type that were introduced by P. Ramond, by A. Neveu and J.H. Schwarz, and by others. If one separates the two coordinates  $\sigma^a = (\sigma^0, \sigma^1)$  into a temporal coordinate,  $\tau \equiv \sigma^0$ , and a spatial coordinate,  $\sigma \equiv \sigma^1$ , the configuration “at time  $\tau$ ” of the string is described by the functions  $X^\mu(\tau, \sigma)$ , where one can interpret  $\sigma$  as a curvilinear abscissa describing the spatial extent of the string. If we consider a closed string, one that is topologically equivalent to a circle, the function  $X^\mu(\tau, \sigma)$  must be periodic in  $\sigma$ . One can show that (modulo the imposition of certain constraints) one can choose the coordinates  $\tau$  and  $\sigma$  on the string such that  $d\Sigma^2 = -d\tau^2 + d\sigma^2$ . Then, the dynamical equations for the string (obtained by minimizing the action (27)) reduce to the standard equation for waves on a string:  $-\partial^2 X^\mu / \partial \tau^2 + \partial^2 X^\mu / \partial \sigma^2 = 0$ . The general solution to this equation describes a superposition of waves travelling along the string in both possible directions:  $X^\mu = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma)$ . If we consider a closed string (one that is topologically equivalent to a circle), these two types of wave are independent

of each other. For an open string (with certain reflection conditions at the endpoints of the string) these two types of waves are connected to each other. Moreover, since the string has a finite length in both cases, one can decompose the left- or right-moving waves  $X_L^\mu(\tau + \sigma)$  or  $X_R^\mu(\tau - \sigma)$  as a Fourier series. For example, for a closed string one may write

$$X^\mu(\tau, \sigma) = X_0^\mu(\tau) + \frac{i}{\sqrt{2}} \ell_s \sum_{n=1}^{\infty} \left( \frac{a_n^\mu}{\sqrt{n}} e^{-2in(\tau-\sigma)} + \frac{\tilde{a}_n^\mu}{\sqrt{n}} e^{-2in(\tau+\sigma)} \right) + \text{h.c.} \quad (28)$$

Here  $X_0^\mu(\tau) = x^\mu + 2\ell_s^2 p^\mu \tau$  describes the motion of the string's center of mass, and the remainder describes the decomposition of the motion around the center of mass into a discrete set of oscillatory modes. Like any vibrating string, a relativistic string can vibrate in its fundamental mode ( $n = 1$ ) or in a “harmonic” of the fundamental mode (for an integer  $n > 1$ ). In the classical case the complex coefficients  $a_n^\mu$ ,  $\tilde{a}_n^\mu$  represent the (complex) amplitudes of vibration for the modes of oscillation at frequency  $n$  times the fundamental frequency. (with  $a_n^\mu$  corresponding to a wave travelling to the right, while  $\tilde{a}_n^\mu$  corresponds to a wave travelling to the left.) When one quantizes the string dynamics the position of the string  $X^\mu(\tau, \sigma)$  becomes an operator (acting in the space of quantum states of the system), and because of this the quantities  $x^\mu$ ,  $p^\mu$ ,  $a_n^\mu$  and  $\tilde{a}_n^\mu$  in (28) become operators. The notation h.c. signifies that one must add the hermitian conjugates of the oscillation terms, which will contain the operators  $(a_n^\mu)^\dagger$  and  $(\tilde{a}_n^\mu)^\dagger$ . (The notation  $\dagger$  indicates hermitian conjugation, in other words the operator analog of complex conjugation.) One then finds that the operators  $x^\mu$  and  $p^\mu$  describing the motion of the center of mass satisfy the usual commutation relations of a relativistic particle,  $[x^\mu, p^\mu] = i\hbar \eta^{\mu\nu}$ , and that the operators  $a_n^\mu$  and  $\tilde{a}_n^\mu$  become annihilation operators, like those that appear in the quantum theory of any vibrating system:  $[a_n^\mu, (a_m^\nu)^\dagger] = \hbar \eta^{\mu\nu} \delta_{nm}$ ,  $[\tilde{a}_n^\mu, (\tilde{a}_m^\nu)^\dagger] = \hbar \eta^{\mu\nu} \delta_{mn}$ . In the case of an open string, one only has *one* set of oscillators, let us say  $a_n^\mu$ . The discussion up until now has neglected to mention that the oscillation amplitudes  $a_n^\mu$ ,  $\tilde{a}_n^\mu$  must satisfy an infinite number of constraints (connected with the equation obtained by minimizing (27) with respect to the auxiliary metric  $\gamma_{ab}$ ). One can satisfy these by expressing *two* of the space-time components of the oscillators  $a_n^\mu$ ,  $\tilde{a}_n^\mu$  (for each  $n$ ) as a function of the other. Because of this, the physical states of the string are described by oscillators  $a_n^i$ ,  $\tilde{a}_n^i$  where the index  $i$  only takes  $D - 2$  values in a space-time of dimension  $D$ . Forgetting this subtlety for the moment (which is nevertheless crucial physically), let us conclude this discussion by summarizing the *spectrum* of a quantum string, or in other words the ensemble of quantum states of motion for a string.

For an open string, the ensemble of quantum states describes the states of motion (the momenta  $p^\mu$ ) of an infinite collection of relativistic particles, having squared masses  $M^2 = -\eta_{\mu\nu} p^\mu p^\nu$  equal to  $(N - 1) m_s^2$ , where  $N$  is a non-negative integer and  $m_s \equiv \hbar/\ell_s$  is the fundamental mass of string theory associated to the fundamental length  $\ell_s$ . For a closed string, one finds another “infinite tower” of more and more massive particles, this time with  $M^2 = 4(N - 1) m_s^2$ . In both cases the integer  $N$  is given, as a function of the string's oscillation amplitudes

(travelling to the right), by

$$N = \sum_{n=1}^{\infty} n \eta_{\mu\nu} (a_n^\mu)^\dagger a_n^\nu. \quad (29)$$

In the case of a closed string one must also satisfy the constraint  $N = \tilde{N}$  where  $\tilde{N}$  is the operator obtained by replacing  $a_n^\mu$  by  $\tilde{a}_n^\mu$  in (29).

The preceding result essentially states that the (quantized) internal energy of an oscillating string defines the squared mass of the associated particle. The presence of the additional term  $-1$  in the formulae given above for  $M^2$  means that the quantum state of minimum internal energy for a string, that is, the “vacuum” state  $|0\rangle$  where all oscillators are in their ground state,  $a_n^\mu |0\rangle = 0$ , corresponds to a negative squared mass ( $M^2 = -m_s^2$  for the open string and  $M^2 = -4m_s^2$  for the closed string). This unusual quantum state (a *tachyon*) corresponds to an instability of the theory of bosonic strings. It is absent from the more sophisticated versions of string theory (“superstrings”) due to F. Gliozzi, J. Scherk, and D. Olive, to M. Green and J.H. Schwarz, and to D. Gross and collaborators. Let us concentrate on the other states (which are the only ones that have corresponding states in superstring theory). One then finds that the first possible physical quantum states (such that  $N = 1$ ) describe some massless particles. In relativistic quantum theory it is known that any particle is the quantized excitation of a corresponding field. Therefore the massless particles that appear in string theory must correspond to long-range fields. To know which fields appear in this way one must more closely examine which possible combinations of oscillator excitations  $a_1^\mu, a_2^\mu, a_3^\mu, \dots$ , appearing in Formula (29), can lead to  $N = 1$ . Because of the factor  $n$  in (29) multiplying the harmonic contribution of order  $n$  to the mass squared, only the oscillators of the fundamental mode  $n = 1$  can give  $N = 1$ . One then deduces that the internal quantum states of massless particles appearing in the theory of *open strings* are described by a string oscillation state of the form

$$\zeta_\mu (a_1^\mu)^\dagger |0\rangle. \quad (30)$$

On the other hand, because of the constraint  $N = \tilde{N} = 1$ , the internal quantum states of the massless particles appearing in the theory of *closed strings* are described by a state of excitation containing both a left-moving oscillation and a right-moving oscillation:

$$\zeta_{\mu\nu} (a_1^\mu)^\dagger (\tilde{a}_1^\nu)^\dagger |0\rangle. \quad (31)$$

In Equations (30) and (31) the state  $|0\rangle$  denotes the ground state of all oscillators ( $a_n^\mu |0\rangle = \tilde{a}_n^\mu |0\rangle = 0$ ).

The state (30) therefore describes a massless particle (with momentum satisfying  $\eta_{\mu\nu} p^\mu p^\nu = 0$ ), possessing an “internal structure” described by a vector polarization  $\zeta_\mu$ . Here we recognize exactly the definition of a photon, the quantum state associated with a wave  $A_\mu(x) = \zeta_\mu \exp(i k_\lambda x^\lambda)$ , where  $p^\mu = \hbar k^\mu$ . The theory of open strings therefore contains Maxwell’s theory. (One can also

show that, because of the constraints briefly mentioned above, the polarization  $\zeta_\mu$  must be transverse,  $k^\mu \zeta_\mu = 0$ , and that it is only defined up to a gauge transformation:  $\zeta'_\mu = \zeta_\mu + a k_\mu$ . As for the state (31), this describes a massless particle ( $\eta_{\mu\nu} p^\mu p^\nu = 0$ ), possessing an “internal structure” described by a tensor polarization  $\zeta_{\mu\nu}$ . The plane wave associated with such a particle is therefore of the form  $\bar{h}_{\mu\nu}(x) = \zeta_{\mu\nu} \exp(i k_\lambda x^\lambda)$ , where  $p^\mu = \hbar k^\mu$ . As in the case of the open string, one can show that  $\zeta_{\mu\nu}$  must be transverse,  $\zeta_{\mu\nu} k^\nu = 0$  and that it is only defined up to a gauge transformation,  $\zeta'_{\mu\nu} = \zeta_{\mu\nu} + k_\mu a_\nu + k_\nu b_\mu$ . We here see the same type of structure appear that we had in general relativity for plane waves. However, here we have a structure that is richer than that of general relativity. Indeed, since the state (31) is obtained by combining two independent states of oscillation,  $(a_1^\mu)^\dagger$  and  $(\tilde{a}_1^\mu)^\dagger$ , the polarization tensor  $\zeta_{\mu\nu}$  is not constrained to be symmetric. Moreover it is not constrained to have vanishing trace. Therefore, if we decompose  $\zeta_{\mu\nu}$  into its possible irreducible parts (a symmetric traceless part, a symmetric part with trace, and an antisymmetric part) we find that the field  $\bar{h}_{\mu\nu}(x)$  associated with the massless states of a closed string decomposes into: (i) a field  $h_{\mu\nu}(x)$  (the *graviton*) representing a weak gravitational wave in general relativity, (ii) a scalar field  $\Phi(x)$  (called the *dilaton*), and (iii) an antisymmetric tensor field  $B_{\mu\nu}(x) = -B_{\nu\mu}(x)$  subject to the gauge invariance  $B'_{\mu\nu}(x) = B_{\mu\nu}(x) + \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x)$ . Moreover, when one studies the non-linear interactions between these various fields, as described by the transition amplitudes  $A(f, i)$  in string theory, one can show that the field  $h_{\mu\nu}(x)$  truly represents a deformation of the flat geometry of the background space-time in which the theory was initially formulated. Let us emphasize this remarkable result. We started from a theory that studied the quantum dynamics of a string in a rigid background space-time. This theory predicts that *certain quantum excitations of a string* (that propagate at the speed of light) *in fact represent waves of deformation of the space-time geometry*. In intuitive terms, the “elasticity” of space-time postulated by the theory of general relativity appears here as being due to certain internal vibrations of an elastic object extended in one spatial dimension.

Another suggestive consequence of string theory is the link suggested by the comparison between (30) and (31). Roughly, Equation (31) states that the internal state of a closed string corresponding to a graviton is constructed by taking the (tensor) product of the states corresponding to photons in the theory of open strings. This unexpected link between Einstein’s gravitation ( $g_{\mu\nu}$ ) and Maxwell’s theory ( $A_\mu$ ) translates, when we look at interactions in string theory, into remarkable identities (due to H. Kawai, D.C. Lewellen, and S.-H.H. Tye) between the transition amplitudes of open strings and those of closed strings. This affinity between electromagnetism, or rather Yang-Mills theory, and gravitation has recently given rise to fascinating conjectures (due to A. Polyakov and J. Maldacena) connecting quantum Yang-Mills theory in flat space-time to quasi-classical limits of string theory and gravitation in curved space-time. Einstein would certainly have been interested to see how classical general relativity is used here to clarify the limit of a *quantum* Yang-Mills theory.

Having explained the starting point of string theory, we can outline the in-



tuitive reason for which this theory avoids the problems with divergent integrals that appeared when one tried to directly quantize gravitation. We have seen that string theory contains an infinite tower of particles whose masses grow with the degree of excitation of the string's internal oscillators. The gravitational field appears in the limit that one considers the low energy interactions ( $E \ll m_s$ ) between the massless states of the theory. In this limit the graviton (meaning the particle associated with the gravitational field) is treated as a "point-like" particle. When we consider more complicated processes (at one loop,  $\ell = 1$ , see above), virtual elementary gravitons could appear with arbitrarily high energy. It is these virtual high-energy gravitons that are responsible for the divergences. However, in string theory, when we consider any intermediate process whatsoever where high energies appear, it must be remembered that this high intermediate energy can also be used to excite the internal state of the virtual gravitons, and thus reveal that they are "made" from an extended string. An analysis of this fact shows that string theory introduces an effective truncation of the type  $E \lesssim m_s$  on the energies of exchanged virtual particles. In other words, the fact that there are no truly "point-like" particles in string theory, but only string excitations having a characteristic length  $\sim \ell_s$ , eliminates the problem of infinities connected to arbitrarily small length and time scales. Because of this, in string theory one can calculate the transition amplitudes corresponding to a collision between two gravitons, and one finds that the result is given by a finite integral [25].

Up until now we have only considered the starting point of string theory. This is a complex theory that is still in a stage of rapid development. Let us briefly sketch some other aspects of this theory that are relevant for this exposé centered around relativistic gravitation. Let us first state that the more sophisticated versions of string theory (*superstrings*) require the inclusion of fermionic oscillators  $b_n^\mu, \tilde{b}_n^\mu$ , in addition to the bosonic oscillators  $a_n^\mu, \tilde{a}_n^\mu$  introduced above. One then finds that there are no particles of negative mass-squared, and that the space-time dimension  $D$  must be equal to 10. One also finds that the massless states contain more states than those indicated above. In fact, one finds that the fields corresponding to these states describe the various possible theories of supergravity in  $D = 10$ . Recently (in work by J. Polchinski) it has also been understood that string theory contains not only the states of excitation of strings (in other words of objects extended in one spatial direction), but also the states of excitation of objects extended in  $p$  spatial directions, where the integer  $p$  can take other values than 1. For example,  $p = 2$  corresponds to a *membrane*. It even seems (according to C. Hull and P. Townsend) that one should recognize that there is a sort of "democracy" between several different values for  $p$ . An object extended in  $p$  spatial directions is called a *p-brane*. In general, the masses of the quantum states of these  $p$ -branes are very large, being parametrically higher than the characteristic mass  $m_s$ . However, one may also consider a limit where the mass of certain  $p$ -branes tends towards zero. In this limit, the fields associated with these  $p$ -branes become long-range fields. A surprising result (by E. Witten) is that, in this limit, the infinite tower of states of certain  $p$ -branes (in particular for  $p = 0$ ) corresponds exactly to the infinite

tower of states that appear when one considers the maximal supergravity in  $D = 11$  dimensions, with the eleventh (spatial) dimension compactified on a circle (that is to say with a periodicity condition on  $x^{11}$ ). In other words, in a certain limit, a theory of superstrings in  $D = 10$  transforms into a theory that lives in  $D = 11$  dimensions! Because of this, many experts in string theory believe that the true definition of string theory (which is still to be found) must start from a theory (to be defined) in 11 dimensions (known as “ $M$ -theory”).

We have seen in Section 8 that one point of contact between relativistic gravitation and quantum theory is the phenomenon of thermal emission from black holes discovered by S.W. Hawking. String theory has shed new light upon this phenomenon, as well as on the concept of black hole “entropy.” The essential question that the calculation of S.W. Hawking left in the shadows is: what is the physical meaning of the quantity  $S$  defined by Equation (19)? In the thermodynamic theory of ordinary bodies, the entropy of a system is interpreted, since Boltzmann’s work, as the (natural) logarithm of the number of microscopic states  $N$  having the same macroscopic characteristics (energy, volume, etc.) as the state of the system under consideration:  $S = \log N$ . Bekenstein had attempted to estimate the number of microscopic internal states of a macroscopically defined black hole, and had argued for a result such that  $\log N$  was on the order of magnitude of  $A/\hbar G$ , but his arguments remained indirect and did not allow a clear meaning to be attributed to this counting of microscopic states. Work by A. Sen and by A. Strominger and C. Vafa, as well as by C.G. Callan and J.M. Maldacena has, for the first time, given examples of black holes whose microscopic description in string theory is sufficiently precise to allow for the calculation (in certain limits) of the number of internal quantum states,  $N$ . It is therefore quite satisfying to find a final result for  $N$  whose logarithm is *precisely* equal to the expression (19). However, there do remain dark areas in the understanding of the quantum structure of black holes. In particular, the string theory calculations allowing one to give a precise statistical meaning to the entropy (19) deal with very special black holes (known as *extremal* black holes, which have the maximal electric charge that a black hole with a regular horizon can support). These black holes have a Hawking temperature equal to zero, and therefore do not emit thermal radiation. They correspond to *stable* states in the quantum theory. One would nevertheless also like to understand the detailed internal quantum structure of *unstable* black holes, such as the Schwarzschild black hole (17), which has a non-zero temperature, and which therefore loses its mass little by little in the form of thermal radiation. What is the final state to which this gradual process of black hole “evaporation” leads? Is it the case that an initial pure quantum state radiates all of its initial mass to transform itself entirely into incoherent thermal radiation? Or does a Schwarzschild black hole transform itself, after having obtained a minimum size, into something else? The answers to these questions remain open to a large extent, although it has been argued that a Schwarzschild black hole transforms itself into a highly massive quantum string state when its radius becomes on the order of  $\ell_s$  [26].

We have seen previously that string theory contains general relativity in a certain limit. At the same time, string theory is, strictly speaking, infinitely

richer than Einstein's gravitation, for the graviton is nothing more than a particular quantum excitation of a string, among an infinite number of others. What deviations from Einstein's gravity are predicted by string theory? This question remains open today because of our lack of comprehension about the connection between string theory and the reality observed in our everyday environment (4-dimensional space-time; electromagnetic, weak, and strong interactions; the spectrum of observed particles; ...). We shall content ourselves here with outlining a few possibilities. (See the contribution by I. Antoniadis for a discussion of other possibilities.) First, let us state that if one considers collisions between gravitons with energy-momentum  $k$  smaller than, but not negligible with respect to, the characteristic string mass  $m_s$ , the calculations of transition amplitudes in string theory show that the usual Einstein equations (in the absence of matter)  $R_{\mu\nu} = 0$  must be modified, by including corrections of order  $(k/m_s)^2$ . One finds that these modified Einstein equations have the form (for bosonic string theory)

$$R_{\mu\nu} + \frac{1}{4} \ell_s^2 R_{\mu\alpha\beta\gamma} R_{\nu}^{\alpha\beta\gamma} + \dots = 0, \quad (32)$$

where

$$R_{\nu\alpha\beta}^{\mu} \equiv \partial_{\alpha} \Gamma_{\nu\beta}^{\mu} + \Gamma_{\sigma\alpha}^{\mu} \Gamma_{\nu\beta}^{\sigma} - \partial_{\beta} \Gamma_{\nu\alpha}^{\mu} - \Gamma_{\sigma\beta}^{\mu} \Gamma_{\nu\alpha}^{\sigma}, \quad (33)$$

denotes the "curvature tensor" of the metric  $g_{\mu\nu}$ . (the quantity  $R_{\mu\nu}$  defined in Section 5 that appears in Einstein's equations in an essential way is a "trace" of this tensor:  $R_{\mu\nu} = R_{\mu\sigma\nu}^{\sigma}$ .) As indicated by the dots in (32), the terms written are no more than the two first terms of an infinite series in growing powers of  $\ell_s^2 \equiv \alpha'$ . Equation (32) shows how the fact that the string is not a point, but is rather extended over a characteristic length  $\sim \ell_s$ , modifies the Einsteinian description of gravity. The corrections to Einstein's equation shown in (32) are nevertheless completely negligible in most applications of general relativity. In fact, it is expected that  $\ell_s$  is on the order of the Planck scale  $\ell_p$ , Equation (25). More precisely, one expects that  $\ell_s$  is on the order of magnitude of  $10^{-32}$  cm. (Nevertheless, this question remains open, and it has been recently suggested that  $\ell_s$  is much larger, and perhaps on the order of  $10^{-17}$  cm.)

If one assumes that  $\ell_s$  is on the order of magnitude of  $10^{-32}$  cm (and that the extra dimensions are compactified on distances scales on the order of  $\ell_s$ ), the only area of general relativistic applications where the modifications shown in (32) should play an important role is in primordial cosmology. Indeed, close to the initial singularity of the Big Bang (if it exists), the "curvature"  $R_{\mu\nu\alpha\beta}$  becomes extremely large. When it reaches values comparable to  $\ell_s^{-2}$  the infinite series of corrections in (32) begins to play a role comparable to the first term, discovered by Einstein. Such a situation is also found in the interior of a black hole, when one gets very close to the singularity (see Figure 3). Unfortunately, in such situations, one must take the infinite series of terms in (32) into account, or in other words replace Einstein's description of gravitation in terms of a *field* (which corresponds to a *point-like* (quantum) particle) by its exact stringy description. This is a difficult problem that no one really knows how to attack today.

However, a priori string theory predicts more drastic low energy ( $k \ll m_s$ ) modifications to general relativity than the corrections shown in (32). In fact, we have seen in Equation (31) above that Einsteinian gravity does not appear alone in string theory. It is always necessarily accompanied by other long-range fields, in particular a scalar field  $\Phi(x)$ , the *dilaton*, and an antisymmetric tensor  $B_{\mu\nu}(x)$ . What role do these “partners” of the graviton play in observable reality? This question does not yet have a clear answer. Moreover, if one recalls that (super)string theory must live in a space-time of dimension  $D = 10$ , and that it includes the  $D = 10$  (and eventually the  $D = 11$ ) theory of supergravity, there are many other supplementary fields that add themselves to the ten components of the usual metric tensor  $g_{\mu\nu}$  (in  $D = 4$ ). It is conceivable that all of these supplementary fields (which are massless to first approximation in string theory) acquire masses in our local universe that are large enough that they no longer propagate observable effects over macroscopic scales. It remains possible, however, that one or several of these fields remain (essentially) massless, and therefore can propagate physical effects over distances that are large enough to be observable. It is therefore of interest to understand what physical effects are implied, for example, by the dilaton  $\Phi(x)$  or by  $B_{\mu\nu}(x)$ . Concerning the latter, it is interesting to note that (as emphasized by A. Connes, M. Douglas, and A. Schwartz), in a certain limit, the presence of a background  $B_{\mu\nu}(x)$  has the effect of deforming the space-time geometry in a “non-commutative” way. This means that, in a certain sense, the space-time coordinates  $x^\mu$  cease to be simple real (commuting) numbers in order to become non-commuting quantities:  $x^\mu x^\nu - x^\nu x^\mu = \varepsilon^{\mu\nu}$  where  $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$  is connected to a (uniform) background  $B_{\mu\nu}$ . To conclude, let us consider the other obligatory partner of the graviton  $g_{\mu\nu}(x)$ , the dilaton  $\Phi(x)$ . This field plays a central role in string theory. In fact, the average value of the dilaton (in the vacuum) determines the string theory coupling constant,  $g_s = e^\Phi$ . The value of  $g_s$  in turn determines (along with other fields) the physical coupling constants. For example, the gravitational coupling constant is given by a formula of the type  $\hbar G = \ell_s^2(g_s^2 + \dots)$  where the dots denote correction terms (which can become quite important if  $g_s$  is not very small). Similarly, the fine structure constant,  $\alpha = e^2/\hbar c \simeq 1/137$ , which determines the intensity of electromagnetic interactions is a function of  $g_s^2$ . Because of these relations between the physical coupling constants and  $g_s$  (and therefore the value of the dilaton;  $g_s = e^\Phi$ ), we see that if the dilaton is massless (or in other words is long-range), its value  $\Phi(x)$  at a space-time point  $x$  will depend on the distribution of matter in the universe. For example, as is the case with the gravitational field (for example  $g_{00}(x) \simeq -1 + 2GM/c^2 r$ ), we expect that the value of  $\Phi(x)$  depends on the masses present around the point  $x$ , and should be different at the Earth’s surface than it is at a higher altitude. One may also expect that  $\Phi(x)$  would be sensitive to the expansion of the universe and would vary over a time scale comparable to the age of the universe. However, if  $\Phi(x)$  varies over space and/or time, one concludes from the relations shown above between  $g_s = e^\Phi$  and the physical coupling constants that the latter must also vary over space and/or time. Therefore, for example, the value, here and now, of the fine structure constant  $\alpha$  could be slightly different from the value

it had, long ago, in a very distant galaxy. Such effects are accessible to detailed astronomical observations and, in fact, some recent observations have suggested that the interaction constants were different in distant galaxies. However, other experimental data (such as the fossil nuclear reactor at Oklo and the isotopic composition of ancient terrestrial meteorites) put very severe limits on any variability of the coupling “constants.” Let us finally note that if the fine structure “constant”  $\alpha$ , as well as other coupling “constants,” varies with a massless field such as the dilaton  $\Phi(x)$ , then this implies a violation of the basic postulate of general relativity: the principle of equivalence. In particular, one can show that the universality of free fall is necessarily violated, meaning that bodies with different nuclear composition would fall with different accelerations in an external gravitational field. This gives an important motivation for testing the principle of equivalence with greater precision. For example, the MICROSCOPE space mission [27] (of the CNES) should soon test the universality of free fall to the level of  $10^{-15}$ , and the STEP space project (Satellite Test of the Equivalence Principle) [28] could reach the level  $10^{-18}$ .

Another interesting phenomenological possibility is that the dilaton (and/or other scalar fields of the same type, called *moduli*) acquires a non-zero mass that is however very small with respect to the string mass scale  $m_s$ . One could then observe a modification of Newtonian gravitation over small distances (smaller than a tenth of a millimeter). For a discussion of this theoretical possibility and of its recent experimental tests see, respectively, the contributions by I. Antoniadis and J. Mester to this Poincaré seminar.

## 12 Conclusion

For a long time general relativity was admired as a marvellous intellectual construction, but it only played a marginal role in physics. Typical of the appraisal of this theory is the comment by Max Born [29] made upon the fiftieth anniversary of the *annus mirabilis*: “The foundations of general relativity seemed to me then, and they still do today, to be the greatest feat of human thought concerning Nature, the most astounding association of philosophical penetration, physical intuition, and mathematical ability. However its connections to experiment were tenuous. It seduced me like a great work of art that should be appreciated and admired from a distance.”

Today, one century after the *annus mirabilis*, the situation is quite different. General relativity plays a central role in a large domain of physics, including everything from primordial cosmology and the physics of black holes to the observation of binary pulsars and the definition of international atomic time. It even has everyday practical applications, via the satellite positioning systems (such as the GPS and, soon, its European counterpart Galileo). Many ambitious (and costly) experimental projects aim to test it (G.P.B., MICROSCOPE, STEP, ...), or use it as a tool for deciphering the distant universe (LIGO/VIRGO/GEO, LISA, ...). The time is therefore long-gone that its connection with experiment was tenuous. Nevertheless, it is worth noting that the

fascination with the structure and physical implications of the theory evoked by Born remains intact. One of the motivations for thinking that the theory of strings (and other extended objects) holds the key to the problem of the unification of physics is its deep affinity with general relativity. Indeed, while the attempts at “Grand Unification” made in the 1970s completely ignored the gravitational interaction, string theory necessarily leads to Einstein’s fundamental concept of a dynamical space-time. At any rate, it seems that one must more deeply understand the “generalized quantum geometry” created through the interaction of strings and  $p$ -branes in order to completely formulate this theory and to understand its hidden symmetries and physical implications. Einstein would no doubt appreciate seeing the key role played by symmetry principles and gravity within modern physics.

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